An Introduction to Quantum Computation in Geometric Algebra

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Geometric Algebra

What is it ? Why using it ?

What is Geometric Algebra (GA)

Geometric

Interpretation and interaction using vectors.

Algebra

A vector space closed under multiplication.

What is Geometric Algebra

Geometric algebra (GA) is a powerful tool in giving algebraic structures their geometric interpretations.

English mathematician William Kingdon Clifford (1882) first proposed geometric algebra, but its importance wasn't recognized then.

Until American physicist David Hestenes applied geometric algebra to fields of calculus, physics, and much more, the power and elegance of geometric algebra thus shown.

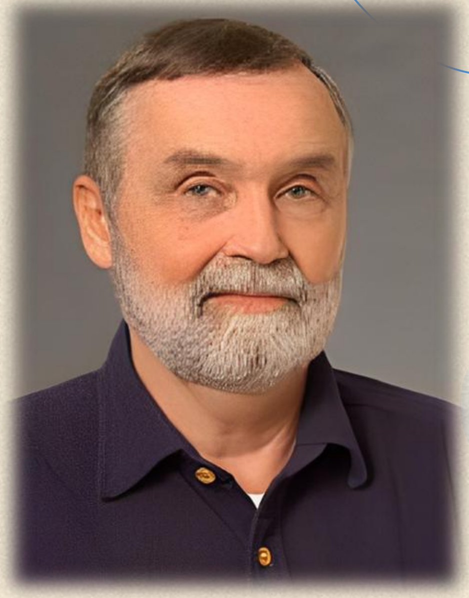
William Kingdon Clifford

Why use Geometric Algebra

Geometric algebra provides us with insights in classical mechanics, relativity, the theory of quantum mechanic, and much more.

Hence, it is of no surprise that it can be utilized in the theory of quantum information and computation as well.

Our goal is to dig out the simplicity
and intuition of mathmetical operations
inside QIC from a new perspective.



David Hestenes

GA Fundamentals

The vector space considered in GA is a Euclidean space. we have vectors as before, and operations such as dot product are defined in the same way.

What's new is that GA introduces wedge product and geometric product.

Consider two vectors a and b, we have :

$$\text{dot product of } a, b \equiv a \cdot b$$

$$\text{wedge product of } a, b \equiv a \wedge b$$

$$\text{geometric product of } a, b \equiv ab = a \cdot b + a \wedge b$$

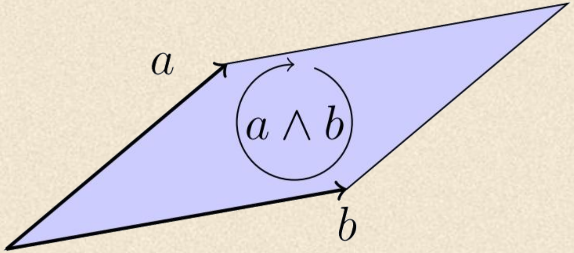
Wedge Product

$$|a \wedge b| \coloneqq |a| \times |b| \times sin(\theta)\,,$$

where θ is the angle between two vectors a and b.

However, the result of doing $a \wedge b$ is that a bivector is generated, which could be taken as the "oriented area" shown below; similarly, a trivector $a \wedge b \wedge c$ represents a "oriented volume" and so on.

Generally, such kind of vectors in various dimension are called multivectors.



Geometric Product

Recall that :

$$\text{geometric product of } a, b \equiv ab = a \cdot b + a \wedge b$$

For parallel vectors

$$ab = a \cdot b + a \wedge b = a \cdot b = ba$$

For perpendicular vectors

$$ab = a \cdot b + a \wedge b = a \wedge b = -ba$$

Geometric Algebra for $\mathbb{R}^n$

For an n-dim Euclidean space $\mathbb{R}^n$ with orthonormal basis $\{e_i\}$:

$$e_i \wedge e_j \wedge \cdots \wedge e_k = e_i e_j \cdots e_k$$

After collecting all such multivector and create a new space $\mathcal{G}_n$, the basis of $\mathcal{G}_n$ would then be :

$$\underbrace{1}_{\text{0-grade}}\,,\quad \overbrace{\mathrm{e}_i(\forall i)}^{\text{1-grade}}\,,\quad \underbrace{\mathrm{e}_i\mathrm{e}_j(\forall i<j)}_{\text{2-grade}}\,,\quad \overbrace{\mathrm{e}_i\mathrm{e}_j\mathrm{e}_k(\forall i<j<k)}^{\text{3-grade}}\,,\quad \cdots$$

$\{e_i\}$ generates multivectors of different grades. There are $\binom{n}{r}$ components in each subspace $\mathcal{G}_n^r$ of grade r.

Grade-Projection Operator

The grade-projection operator

$$\langle\cdot\rangle_r$$

returns the r-grade element of the input.

For example,

$$A = \langle A\rangle_0 + \langle A\rangle_1 + \cdots = \sum_r \langle A\rangle_r$$

By convention,

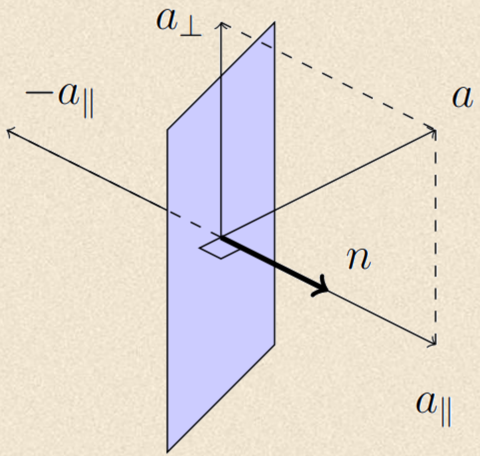
$$\langle A\rangle \equiv \langle A\rangle_0$$

Then we introduce two other operations on multivectors :

reflection and rotation

Reflection

The beauty of geometric algebra lies in its geometric interpretation. First, let us look at reflection of a vector a across the direction of a unit vector n :



$$\begin{aligned}\boldsymbol{a}_{\mathbf{reflect}} &= a - 2a_\parallel = a - 2(a \cdot n)n \\ &= a - (an + na)n \\ &= a - ann - nan \\ &= -\boldsymbol{nan}\end{aligned}$$

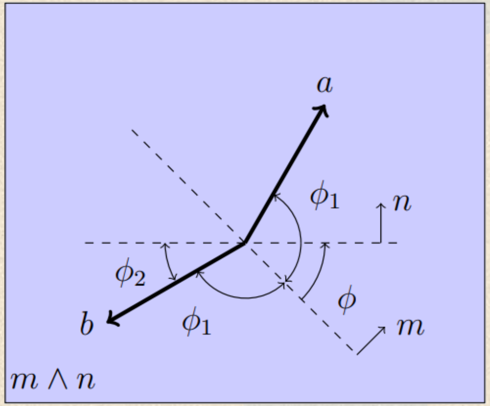
“ Two consecutive reflections make up a rotation.”

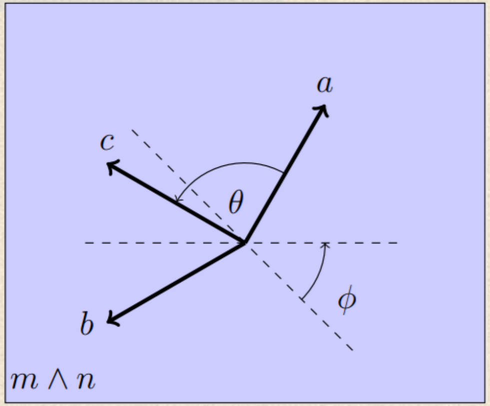
Continues the beauty of geometric algebra.

Rotation

As shown in figure below, the vector a is first reflected along the unit vector m to obtain b, then b is reflected along the unit vector n to obtain c.

The rotated angle $\theta = 2\phi$.



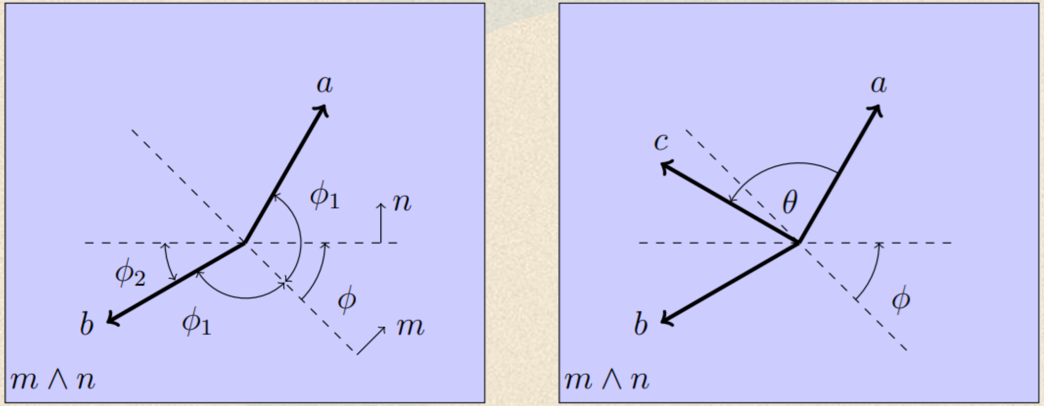


Rotation

Therefore we have :

$$c = -n(-mam)n = (nm)a(mn) = Ra\tilde{R}$$

Such R is called a rotor, and $\tilde{R}$ is the reversion of R with $R\tilde{R} = 1$.



02

Qubit

Introduce qubit system under geometric algebra framework.

STA, the Spacetime Algebra

STA is a branch of geometric algebra, used to describe spacetime physics.
There are four generators for STA, denoted as $\gamma_0, \gamma_1, \gamma_2, \gamma_3$, where :

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$$

We will use STA^+, the even-grade subalgebra in STA, to describe quantum states.
The generators of STA^+ denoted as $\sigma_1, \sigma_2, \sigma_3$ are defined as follow :

$$\sigma_1 \equiv \gamma_1\gamma_0 \,, \quad \sigma_2 \equiv \gamma_2\gamma_0 \,, \quad \sigma_3 \equiv \gamma_3\gamma_0$$

Algebra for Quantum States

We could construct basis of the form :

$1 \times \text{grade } 0$		1	
$3 \times \text{grade } 1$	σ_1	σ_2	σ_3
$3 \times \text{grade } 2$	$I\sigma_1$	$I\sigma_2$	$I\sigma_3$
$1 \times \text{grade } 3$		$I \equiv \sigma_1\sigma_2\sigma_3$	

Note that I is called a pseudoscalar, which commutes with all multivectors in this space and owns the property :

$$II = (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3) = -1$$

Algebra for Quantum States

The basis is actually isomorphic to the **Pauli matrices**:

$$(\sigma_1)^2 = (\sigma_2)^2 = (\sigma_3)^2 = 1$$

$$I = \sigma_1\sigma_2\sigma_3$$

The 2-grade elements satisfy

$$I\sigma_1 = \sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3$$

$$I\sigma_2 = \sigma_1\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_1$$

$$I\sigma_3 = \sigma_1\sigma_2\sigma_3\sigma_3 = \sigma_1\sigma_2$$

Same as Pauli matrices if I is the unit imaginary i.

Pauli matrices $\{\hat{\sigma}_i\}$ are isomorphic to $\{\sigma_i\}$, hence the notation.

Reversion

However, the reversion will be :

$$(a + b\sigma_k + cI\sigma_j + dI)^{\sim} = a - b\sigma_k - cI\sigma_j + dI$$

Due to this specialty, the change in representation from matrix to geometric algebra should be handled with much care.

Adjoint Operation

Let ψ be an n-particle state, the adjoint of ψ is defined as :

$$\psi^\dagger \equiv \left(\prod_{i=1}^{n} \gamma_0^i\right) \tilde{\psi} \left(\prod_{i=1}^{n} \gamma_0^i\right)$$

The notation is “dagger” since it is the same as the Hermitian conjugate in matrix representation, satisfying :

$$\left(a + b\sigma_k + cI\sigma_j + dI\right)^\dagger = a + b\sigma_k - cI\sigma_j - dI$$

$$(\psi\phi)^\dagger = \phi^\dagger\psi^\dagger$$

Single-Qubit State

Proceeding on with the already-developed notations by Hestenes and others, a single-qubit state can be represented as :

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{bmatrix} \overset{GA}{\mapsto} \psi = a^0 + a^k I\sigma_k$$

We can also rewrite the GA form as :

$$\psi = \underbrace{1(a^0 + I\sigma_3 a^3)}_{like\ \psi_1} + \underbrace{(-I\sigma_2)(-a^2 + I\sigma_3 a^1)}_{like\ \psi_2}$$

Observation (1/2)

It appears that :

$$\begin{cases} \text{unit imaginary } i \\ \text{basis of } \psi_1 \\ \text{basis of } \psi_2 \end{cases} \overset{GA}{\mapsto} \begin{cases} I\sigma_3 \\ 1 \\ (-I\sigma_2) \end{cases}$$

We can see that a state ψ is an element in the even-grade subalgebra of $\mathcal{G}_3$, denoted as $\mathcal{G}_3^+$.

We no longer need imaginary numbers !

Observation (2/2)

Some basic examples are :

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{GA}{\mapsto} 1\,, \qquad |+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \overset{GA}{\mapsto} \frac{1}{\sqrt{2}}(1 - I\sigma_2)$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overset{GA}{\mapsto} -I\sigma_2\,, \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \overset{GA}{\mapsto} \frac{1}{\sqrt{2}}(1 + I\sigma_2)$$

The set $\{1, I\sigma_1, I\sigma_2, I\sigma_3\}$ are the conventional basis corresponding to a two-dimensional complex Hilbert space $\mathcal{H}_2$ with computational basis $\{|0\rangle, |1\rangle\}$.

Bloch Vector (1/3)

The Bloch sphere representation of a qubit is :

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\phi} sin\frac{\theta}{2}|1\rangle = \begin{bmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2} cos\phi + i\, sin\frac{\theta}{2} sin\phi \end{bmatrix}$$

Geometric algebra version :

$$\psi = cos\frac{\theta}{2} + sin\frac{\theta}{2}(sin\phi\, I\sigma_1 - cos\phi\, I\sigma_2)$$

Bloch Vector (2/3)

$$\psi = cos\frac{\theta}{2} + sin\frac{\theta}{2}(sin\phi\ I\sigma_1 - cos\phi\ I\sigma_2)$$

Observe that :

$$-(sin\phi\ I\sigma_1 - cos\phi\ I\sigma_2) = (cos\phi - \sigma_1\sigma_2 sin\phi)I\sigma_2 = I\mathrm{e}^{-\sigma_1\sigma_2\phi}\sigma_2$$

Define the colored part as $-B$, note that :

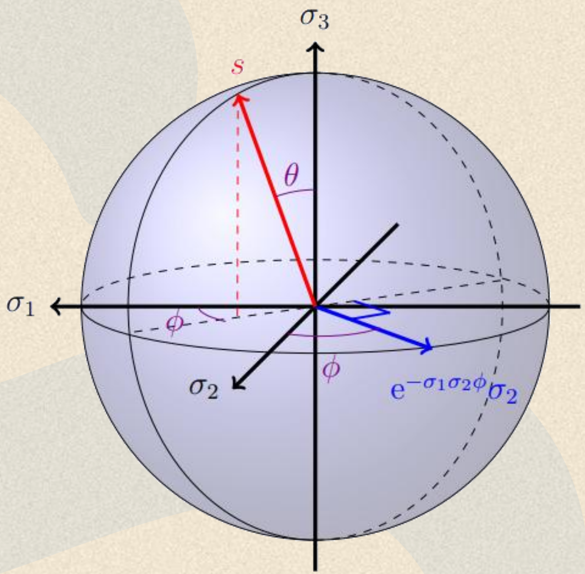
$$B^2 = (-sin\phi\ I\sigma_1 + cos\phi\ I\sigma_2)^2 = -1$$

Hence we could rewrite : Pure states are rotors (spinors)!

$$\psi = cos\frac{\theta}{2} - B\ sin\frac{\theta}{2} = \mathrm{e}^{-B\frac{\theta}{2}} = \mathrm{e}^{-I\sigma_3\frac{\phi}{2}} \cdot \mathrm{e}^{-I\sigma_2\frac{\theta}{2}}$$

Bloch Vector (3/3)

State ψ gives us a spinor (rotor) which rotates σ_3 (or $|0\rangle$ in conventional words) to some vector s on the Bloch sphere, as shown in the plot.



$$\psi\sigma_3\tilde{\psi} = s\,!$$

Multiparticle System (1/2)

To describe an n-particle system, we prepare n copies of the STA^+. For example, we will have n copies of $I\sigma_3$, namely $I\sigma_3^a$ with $a = 1, 2 \cdots, n$.

Bases of different qubits commute under geometric product since :

$$\sigma_k^i \sigma_l^j = \gamma_k^i \gamma_0^i \gamma_l^j \gamma_0^j = (-1)^2 \gamma_l^j \gamma_0^j \gamma_k^i \gamma_0^i = \sigma_l^j \sigma_k^i$$

Therefore all multivectors from different qubits commute, and we will be working in the algebra of :

$$(\mathrm{STA}^+)^{\otimes n} = (\mathrm{STA}^+)^n$$

Multiparticle System (2/2)

Even subalgebra of $(\mathrm{STA}^{+})^{n}$ are of 4^{n} dimensions, but an n-particle state ψ has real dimension $\dim_{\mathbb{R}}(\mathcal{H}_2^n) = 2^{n+1}$ only.

We introduce correlators to fix the issue of redundant dimension.

Details would not be presented here.

$$\text{Correlator } C = \prod_{i=1}^{n} \frac{1}{2}(1 - I^1 I^i)$$

$$\text{State correlator } D = \prod_{i=2}^{n} \frac{1}{2}(1 - I\sigma_3^1 I\sigma_3^i)$$

A Two-qubit Example

$$D = \frac{1}{2}(1 - I\sigma_3^1 I\sigma_3^2)$$

$$|0\rangle \otimes |0\rangle \quad \overset{GA}{\mapsto} \quad (1)(1)D = D$$

$$|0\rangle \otimes |1\rangle \quad \overset{GA}{\mapsto} \quad (1)\left(-I\sigma_2^2\right)D = -I\sigma_2^2 D$$

$$|1\rangle \otimes |0\rangle \quad \overset{GA}{\mapsto} \quad \left(-I\sigma_2^1\right)(1)D = -I\sigma_2^1 D$$

$$|1\rangle \otimes |1\rangle \quad \overset{GA}{\mapsto} \quad \left(-I\sigma_2^2\right)\left(-I\sigma_2^2\right)D = I\sigma_2^2 I\sigma_2^2 D$$

Inner Product

$$\langle\psi|\phi\rangle \mapsto \left\langle\tilde{\psi}E_{+}\phi\right\rangle_{Q} = 2^{n-1}\Big(\underset{\text{Real part}}{\underline{\langle\phi D\tilde{\psi}\rangle}} - \underset{\text{Imaginary part}}{\underline{\langle\phi J\tilde{\psi}\rangle J}}\Big)$$

$$E_{+} = \prod_{k}\frac{1+\sigma_3^k}{2}$$

$$D = \prod_{k}\frac{1}{2}\left(1 - I\sigma_3^1 I\sigma_3^k\right)$$

$$J = DI\sigma_3^1$$

Density Operator (1/3)

Given the $\mathrm{SU}(2)$ isomorphism :

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{bmatrix}$$

$$\overset{GA}{\mapsto} \quad \psi = a^0 + a^1 I\sigma_1 + a^2 I\sigma_2 + a^3 I\sigma_3$$

$$\overset{SU}{\mapsto} a^0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ia^1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + ia^2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + ia^3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} a^0 + ia^3 & a^2 + ia^1 \\ -a^2 + ia^1 & a^0 - ia^3 \end{bmatrix} = \begin{bmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{bmatrix} \equiv \underline{\Psi}$$

Density Operator (2/3)

Given two qubits $|\psi\rangle$ and $|\phi\rangle$, their outer product is defined as :

$$|\psi\rangle\langle\phi| = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\begin{bmatrix}\phi_1^* & \phi_2^*\end{bmatrix} = \underline{\Psi}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\left(\underline{\Phi}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)^\dagger$$

$$= \underline{\Psi}\,\underline{E_+}\,\underline{\Phi} = \underline{\Psi}\,\frac{1}{2}(\mathbf{I} + \hat{\sigma}_3)\underline{\Phi}^\dagger$$

Note that $E_+(\underline{E_+})$ is idempotent and

$$\underline{E_+} \equiv \frac{1 + \hat{\sigma}_3}{2} \equiv \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \overset{GA}{\mapsto} E_+ \equiv \frac{1 + \sigma_3}{2}$$

Density Operator (3/3)

Thus we have :

$$|\psi\rangle\langle\phi| \quad \overset{GA}{\mapsto} \quad \frac{1}{2}\psi(1 + I\sigma_3)\tilde{\phi}$$

When $\psi = \phi$, we have the density operator for a pure state :

$$\rho = \psi\left(\frac{1+\sigma_3}{2}\right)\tilde{\psi} = \psi E_+\tilde{\psi}$$

Gates

03

Operations on single qubit
and multi-qubit states.

Single-Qubit Gate

Pauli Gates, Phase Gate, Hadamard Gate.

Pauli Matrices (1/2)

Pauli matrices $\{\hat{\sigma}_i\}$ are isomorphic to $\{\sigma_i\}$, so we have the notation:

$$\hat{\sigma}_1|\psi\rangle = \begin{bmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{bmatrix} \overset{GA}{\mapsto} \sigma_1(a^0 + a^k I\sigma_k)\sigma_3$$

X Gate

$$\hat{\sigma}_2|\psi\rangle = \begin{bmatrix} a^1 + ia^2 \\ -a^3 + ia^0 \end{bmatrix} \overset{GA}{\mapsto} \sigma_2(a^0 + a^k I\sigma_k)\sigma_3$$

Y Gate

$$\hat{\sigma}_3|\psi\rangle = \begin{bmatrix} a^0 + ia^3 \\ a^2 - ia^1 \end{bmatrix} \overset{GA}{\mapsto} \sigma_3(a^0 + a^k I\sigma_k)\sigma_3$$

Z Gate

Pauli Matrices (2/2)

$$\hat{\sigma}_j|\psi\rangle \quad \overset{GA}{\mapsto} \quad \sigma_j(a^0 + a^k I\sigma_k)\sigma_3$$

The multiplication rule is too complicated, can we simplify it? Think of the geometric interpretation!

$$\psi \mapsto \mathrm{e}^{-I\sigma_j\pi/2}\psi = -I\sigma_j\psi$$

$$\hat{\sigma}_j \overset{GA}{\mapsto} -I\sigma_j$$

The two representations are only off by a global phase.

Imaginary Multiplication

Multiplication by the imaginary unit results in :

$$i|\psi\rangle \quad \overset{GA}{\mapsto} \quad I\psi\sigma_3 = \psi I\sigma_3$$

No need of *i* !

Such multiplication (in geometric algebra) is a sandwich product, so we often denote it as $\mathbf{J} = I\sigma_3$, and :

$$\mathbf{J}\psi = I\psi\sigma_3$$

Phase Gate

The geometric algebra equivalent representation of the phase gate is :

$$\widehat{\mathrm{R}}_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix} \mapsto \mathbf{R}_\alpha = \mathrm{e}^{-\sigma_1\sigma_2\frac{\alpha}{2}}$$

One can check that :

$$\widehat{\mathrm{R}}_\alpha|\psi\rangle = cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i(\phi+\alpha)}sin\frac{\theta}{2}|1\rangle$$

We can obviously see that it is a rotation of angle α on the $\sigma_1\sigma_2$ bivector plane, or equivalently, around the σ_3 axis since $I\sigma_3 = \sigma_1\sigma_2$.

Hadamard Gate (1/2)

We can easily obtain the Hadamard gate as:

$$\hat{\mathrm{H}}|\psi\rangle \quad \overset{GA}{\mapsto} \quad \left(\frac{\hat{\sigma}_1 + \hat{\sigma}_3}{\sqrt{2}}\right)\psi\sigma_3$$

But the multiplication rule can also be changed to

$$H\psi = -I\left(\frac{\hat{\sigma}_1 + \hat{\sigma}_3}{\sqrt{2}}\right)\psi = -I\mathbf{H}\psi$$

Hadamard Gate (2/2)

Therefore :

$$\widehat{\mathrm{H}}\,\hat{\rho}\,\widehat{\mathrm{H}}^{\dagger} \quad \overset{GA}{\mapsto} \quad (-I\mathbf{H})\,\rho\,(-I\mathbf{H})^{\dagger}$$

Furthermore, because **H** is a vector, the result of performing a sandwich product on the density operator is, in fact, a 180-degree rotation about **H**:

$$\mathbf{H} = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)$$

$$H\rho H^{\dagger} = e^{-I\mathbf{H}\frac{\pi}{2}}\,\rho\,e^{I\mathbf{H}\frac{\pi}{2}}$$

Multi-Qubit Gate

Swap Gate, CX Gate, Toffoli Gate, QFT.

Swap Gate

The two-particle swap gate can be easily obtained by the operation of exchanging the raised indices :

$$\mathbf{S}^{(i,j)}[\sigma_k^i] = \mathbf{S}^{(j,i)}\left[\sigma_k^j\right] = \sigma_k^j$$

The [·] denotes applying the gate
on the state inside the bracket

Controlled-NOT Gate (1/2)

Since the CX gate, in matrix notation, can be decomposed as :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{the first particle} = 0} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{the first particle} = 1}$$

Its geometric algebra equivalence is :

$$\mathrm{X}^{2|1} = E_{+}^{1} + E_{-}^{1}\sigma_{1}^{2} \qquad E_{\pm}^{m} = \frac{1 \pm \sigma_{3}^{m}}{2}$$

Controlled-NOT Gate (2/2)

We can check by :

$$\underset{\text{the first particle } = 0}{\underline{\begin{cases} E^1_+[\psi_0] = 1 = \psi_0 \\ E^1_+[\psi_1] = 0 \end{cases}}} \qquad \underset{\text{the first particle } = 1}{\underline{\begin{cases} E^1_-[\psi_0] = 0 \\ E^1_-[\psi_1] = -I\sigma^1_2 = \psi_1 \end{cases}}}$$

When applied onto a density operator ρ, the notation will be far simpler :

$$\rho' = \mathrm{X}^{2|1}\,\rho\,\mathrm{X}^{2|1\,\dagger}$$

Exp. Form of Conditional Gates (1/3)

The conditional operator of the form :

$$U^{2|1} = E_+^1 + E_-^1 U^2$$

where U^2 is a unitary gate on the second system conditioning on the first system satisfying $(U^2)^2 = 1$. It can be rewritten in the exponential form of :

$$U^{2|1} = \exp\left(I(1 - U^2)E_-^1 \frac{\pi}{2}\right)$$

$$\mathrm{X}^{2|1} = E_+^1 + E_-^1 \sigma_1^2 = \exp\left(I\left(1 - \sigma_1^2\right)E_-^1 \frac{\pi}{2}\right)$$

Exp. Form of Conditional Gates (2/3)

The conditional operator of the form :

$$U^{2|1} = E_{+}^{1} + E_{-}^{1} U^{2}$$

where U^2 is a unitary gate on the second system conditioned on the first system satisfying $(U^2)^2 = -1$, can be rewritten in the exponential form of :

$$U^{2|1} = \exp\left(U^2 E_{-}^{1} \frac{\pi}{2}\right)$$

$$\mathrm{X}^{2|1} = E_{+}^{1} - E_{-}^{1} I \sigma_{1}^{2} = \exp\left(-I \sigma_{1}^{2} E_{-}^{1} \frac{\pi}{2}\right)$$

Exp. Form of Conditional Gates (3/3)

If U^2 has an exponential generator G^2 such that

$$U^2 = \exp(G^2)$$

The conditional operator can be rewritten in the exponential form of :

$$U^{2|1} = E_+^1 + E_-^1 U^2 = \exp(G^2 E_-^1)$$

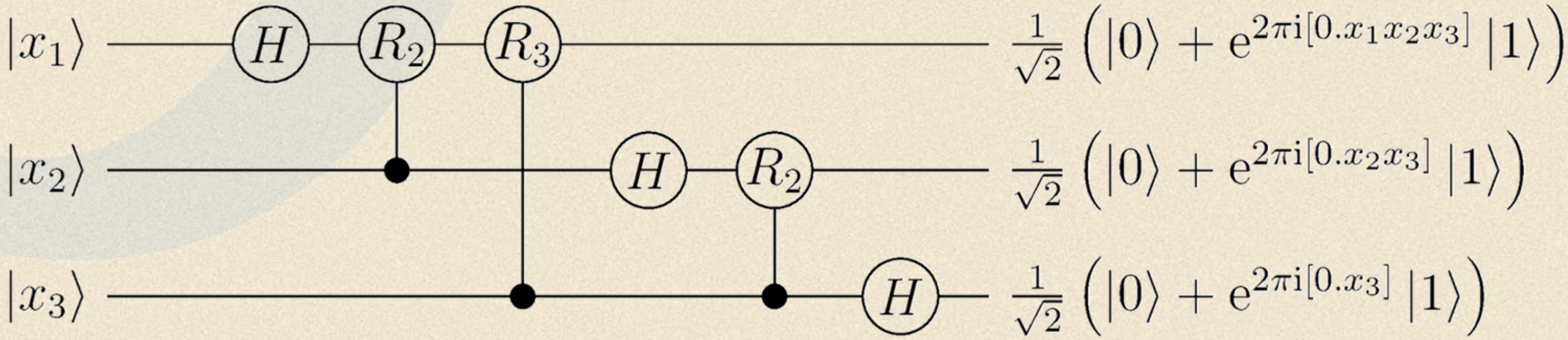
Toffoli Gate

The quantum Toffoli gate takes in three qubit as input, and only reverses the third qubit if the former two are $|1\rangle$'s, else the third is left unchanged.

Hence we can express the Toffoli gate as :

$$\mathrm{X}^{3|1,2} = (\underset{0^1 0^2}{\underline{E_+^1 E_+^2}} + \underset{0^1 1^2}{\underline{E_+^1 E_-^2}} + \underset{1^1 0^2}{\underline{E_-^1 E_+^2}}) + \underset{1^1 1^2}{\underline{E_-^1 E_-^2}}\sigma_1^3 = (1 - E_-^1 E_-^2) + E_-^1 E_-^2 \sigma_1^3$$

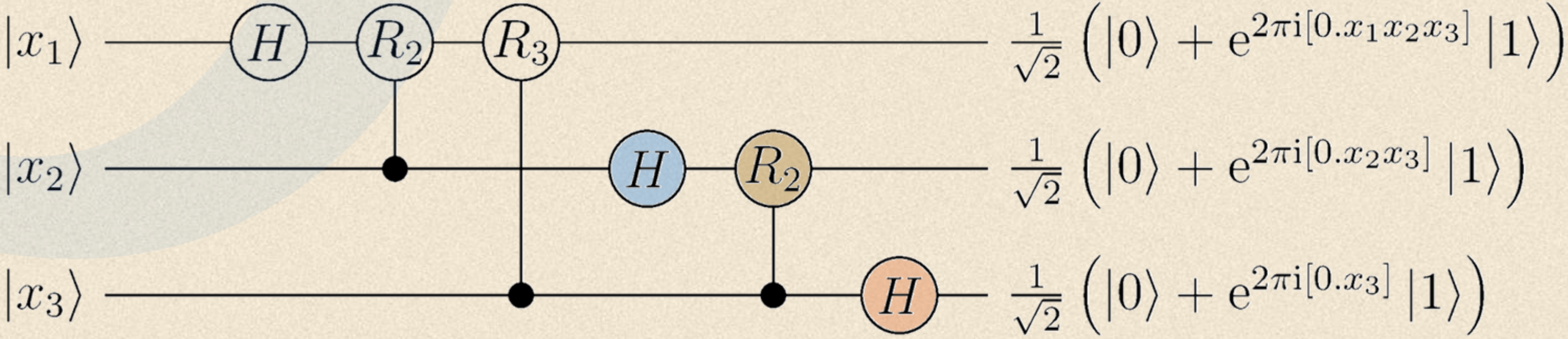
Quantum Fourier Transform (1/4)



Let us define the Hadamard gate and rotation gate of phase $\frac{2\pi}{2^k}$ for particle i as :

$$\mathbf{H}^i = \frac{1}{\sqrt{2}}\left(\sigma_1^i + \sigma_3^i\right) \qquad \mathbf{R}_k^i = e^{-\sigma_1^i\sigma_2^i\frac{2\pi}{2^k}}$$

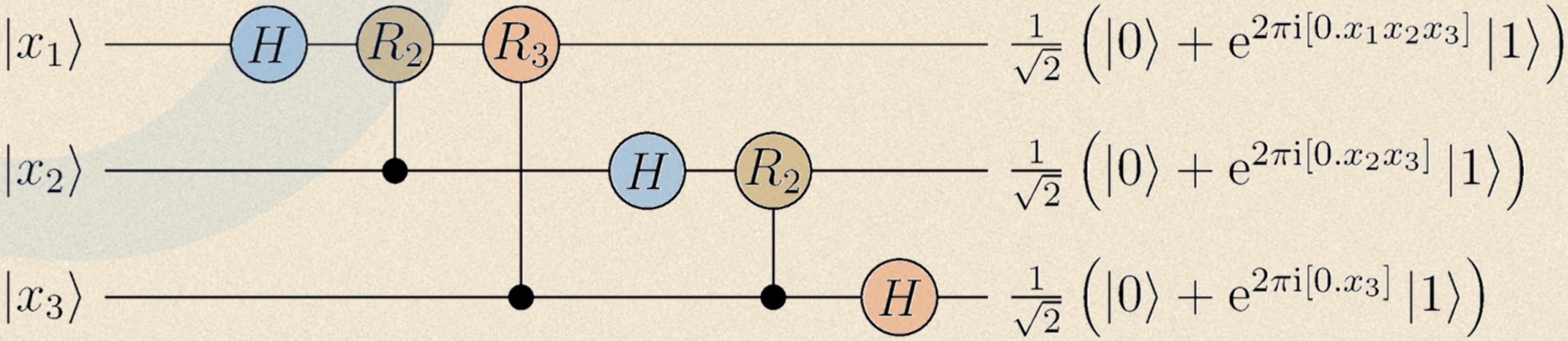
Quantum Fourier Transform (2/4)



Starting from two qubits, observe that the QFT gate is represented as :

$$\mathbf{QFT}_{2^2} = \mathbf{H}^2\left(E_+^2 + E_-^2\mathbf{R}_2^1\right)\mathbf{H}^1$$

Quantum Fourier Transform (3/4)



The QFT gate for three qubits is represented as :

$$\mathbf{QFT}_{2^3} = \mathbf{H}^3\left(E_+^3 + E_-^3\mathbf{R}_2^2\right)\mathbf{H}^2\left(E_+^3 + E_-^3\mathbf{R}_3^1\right)\left(E_+^2 + E_-^2\mathbf{R}_2^1\right)\mathbf{H}^1$$

Quantum Fourier Transform (4/4)

For a general n-qubit QFT gate, it is represented as :

$$\mathbf{QFT}_{2^n} = \mathbf{H}^n$$

$$\left(E_+^n + E_-^n \mathbf{R}_2^{n-1}\right)\mathbf{H}^{n-1}$$

$$\left(E_+^n + E_-^n \mathbf{R}_3^{n-1}\right)\left(E_+^{n-1} + E_-^{n-1} \mathbf{R}_2^{n-2}\right)\mathbf{H}^{n-2}$$

$$\vdots$$

$$\left(E_+^n + E_-^n \mathbf{R}_n^1\right) \cdots \left(E_+^3 + E_-^3 \mathbf{R}_3^1\right)\left(E_+^2 + E_-^2 \mathbf{R}_2^1\right)\mathbf{H}^1$$

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Conclusion

Our findings and what could do more.

Conclusion

Geometric algebra provides us insights from another perspective and gives geometric interpretation to abstract objects. We

1. Introduced the geometric algebra formulation to quantum computation
2. Gave geometric insights to quantum gates
3. Summed up all about the basics of GA-QIC and their notations

With geometric algebra, we may understand more about how quantum system works, looking forward to advance the development of QIC.

Prospect

1. An often used norm: $||M|| = \sqrt{\langle M\widetilde{M}\rangle}$. Can GA provide new definitions of distance measures with geometrical meaning?
2. The unique algebraic structures in wedge product, geometric product may provide new ways in information encoding
3. GA has been utilized in control theory, signal processing and computer graphics → respective developments in quantum theory

That’s all folks!

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