NATIONAL TAIWAN UNIVERSITY
Department of Electrical Engineering
Quantum Information and Computation
Instructor: 鄭皓中教授

An Introduction to Quantum Computation in Geometric Algebra

B09901175　電機三　楊博宇
B09901067　電機三　陳少翔
B10901042　電機二　彭　瑎

Contents

Preface

Why the use of geometric algebra?

Geometric algebra is also called the Clifford algebra. We[1] shall refer to it as geometric algebra, as was coined by its conceiver, the English mathematician William Kingdon Clifford (1882). Its importance was not recognized until promotion by the American physicist David Hestenes, who applied geometric algebra to fields of calculus, physics, and much more.

We follow the notation and formulation created by David Hestenes. By using geometric algebra (spacetime algebra to be exact), both the quantum states and operators live in the same algebra, and the notion of tensor products can be replaced by geometric products. Though this formulation is mathematically equivalent to the Bloch and matrix formulation, it is the notation used that provides deeper (geometric) insight into the quantum objects.

About the sections

Firstly, geometric algebra is not well known for the general researchers; secondly, notations found in literature often contain disparities. Hence, I planned to give the readers a thorough understanding of the material.

- The first section is dedicated to the fundamentals of geometric algebra. If one is already accustomed to its operations and algebraic structure, this chapter can be skipped.
- The second section introduces many of the ideas and notations used in quantum computation using geometric algebra, focusing mainly on how to describe a single quantum binary state - a qubit.
- In the third section, we discuss the mathematical tool needed to represent n-qubit systems - the MSTA, summarizing results in solving the dimensionality inconsistencies between geometric algebra and a quantum system and other miscellaneous operations.
- The forth section organize those about density operators, including operations and measurements on them.

[1] By "we", it meant you the readers and I.

- The fifth section focuses on the geometric picture and intuition on operators and measurements.

Remark. The sections lay down the foundations to study quantum information and computation in the language of geometric algebra. Sections with * on top are there to provide examples, which can be skipped without loss to the understanding of the mathematics.

Geometric algebra on quantum information theory

There have been some attempts to express quantum information theory in GA since 1990s. [HD02; SCH98; Vla99] From the single-qubit to multi-particle operations, the GA approach promises in bringing a more explicit picture to our conception. Although this research field is still very nascent, some works have appeared in some different areas, such as entanglement [Ima08; VTF08], entropy [WW08], no-hiding theorem [Zhu21] and quantum neural networks. [TRF22]

This work sums up several schools of GA notations scattering in different places and aims to provide new insights on quantum information theory based on the new language of geometric algebra. We also provide some of the unproven results seen in literature, aiming for readers to get a full understanding of the computation and ideas.

Task Assignment

楊博宇

- Section 1. Geometric Algebra Fundamentals
- Section 3. Multi-Qubit Systems
- Section 4. Density Operators

陳少翔

- The PowerPoint file for the video presentation.

彭瑎

- Section 1. Geometric Algebra Fundamentals
- Section 2. Single-Qubit Systems
- Section 3. Multi-Qubit Systems
- Section 5. Quantum Gates and Measurements

1 Geometric Algebra Fundamentals

Geometric algebra (GA) is a powerful tool in giving algebraic structures their geometric interpretations. Pioneered by physicist David Hestenes, geometric algebra provides us with insights in classical mechanics, relativity, the theory of quantum mechanic, and much more. It is of no surprise that it can describe the theory of quantum information and computation as well.

1.1 Geometric Product

Here we introduce the fundamentals of geometric algebra[2]. In the Euclidean space of $\mathbb{R}^n$, we denote a set of orthonormal basis as

$$\{\mathsf{e}_i\}\,.$$

We introduce a new product between the vectors of $\mathbb{R}^n$, coined the *geometric product*, denoted as

$$\text{geometric product between vectors } a \text{ and } b := ab.$$

The product is also defined between multiple vectors. The geometric product satisfies the following three axioms between vectors:

1. associativity: $(ab)c = (ab)c = abc$
2. distributivity over vector addition: $a(b+c) = ab + ac$, $(a+b)c = ac + bc$
3. contraction axiom (real sqaures): $aa := a^2 \in \mathbb{R}$.

An *algebra* is a vector space closed under multiplication (in this case, the geometric product). By multiplying multiple vectors together with the geometric product, we obtain *multivectors*, mostly represented by capital italic alphabets, which form the elements of the geometric algebra.

[2] A superb video to introducing geometric algebra is in the reference [sud20].

1.2 Dot and Wedge Product

Observe the following real product (by axiom 3.),

$$(a+b)^2 = a^2 + b^2 + ab + ba.$$

The symmetric sum of $ab + ba$ is a real number, we denote it as the *dot product*:

$$a \cdot b := \frac{1}{2}(ab + ba). \tag{1.1}$$

This is our usual dot product in vector algebra. And the antisymmetric sum is denoted as the *wedge product*:

$$a \wedge b := \frac{1}{2}(ab - ba). \tag{1.2}$$

We can expand the geometric product as

$$ab = a \cdot b + a \wedge b. \tag{1.3}$$

Note that $a \cdot b = b \cdot a$ and $a \wedge b = -b \wedge a$. Hence if a and b are parallel, $ab = a \cdot b$; else if they are orthogonal, $ab = a \wedge b = -ba$. The order of operation between the three products mentioned is: dot product > wedge product > geometric product. This way, many brackets can be dropped hereafter. For example,

$$(a \cdot b)c = a \cdot bc,$$
$$c(a \wedge b) = ca \wedge b,$$
$$(a \cdot b)(c \wedge d) = a \cdot bc \wedge d.$$

Division by a Vector

The geometric product allows for multiplicative inverse[3] of a vector a, namely:

$$a^{-1} \equiv \frac{a}{a^2} \Leftrightarrow a^{-1}a = aa^{-1} = 1. \tag{1.4}$$

[3] The same idea can be extended to all blades with non-zero square.

1.3 Grade

Geometric products between vectors generate objects called multivectors. Given vectors $a_1, a_2, \cdots, a_r$, we define their wedge product as a totally antisymmetric sum

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \frac{1}{r!} \sum (-1)^{\epsilon} a_{k_1} a_{k_2} \cdots a_{k_r}, \tag{1.5}$$

where $\{k_1, \cdots, k_r\}$ is a permutation of $\{1, \cdots, r\}$ and ϵ denotes the number of permutations. If the vectors are linearly-dependent, the product is zero. For a non-zero product of r vectors, this product is called a *blade* of grade r, or an r-grade blade. Any blade of grade r can be represented as the geometric product of r orthogonal vectors, i.e. there exist vectors $e_1, e_2, \cdots, e_r$ such that:

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = e_1 e_2 \cdots e_r.$$

Furthermore, we define the *grade-projection operator* $\langle \cdot \rangle_r$ that returns the r-grade element of the input. A multivector is the sum of multiple blades of different grades, which can hence be decomposed into

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \cdots = \sum_r \langle A \rangle_r.$$

By convention, the 0-grade-projection operator is often abbreviated as

$$\langle A \rangle \equiv \langle A \rangle_0,$$

which is also cyclic-invariant[4]. A multivector A_r is *homogeneous* if it satisfies

$$\langle A_r \rangle_r = A_r,$$

it is often called an r-grade multivector, or simply, an r-vector.

[4] For any multivectors A and B in the same geometric algebra $\mathcal{G}$, we have $\langle AB \rangle = \langle BA \rangle$.

1.4 Geometric Algebra of $\mathbb{R}^n$

For a real Euclidean space $\mathbb{R}^n$, we have a natural orthonormal basis $\{\mathsf{e}_i\}$ (the computational basis) that can be used to generate the whole algebra via the geometric product, denoted as $\mathcal{G}_n$. The bases are:

$$1,\ \ \mathsf{e}_i,\ \ \mathsf{e}_i\mathsf{e}_j\ (i<j),\ \ \mathsf{e}_i\mathsf{e}_j\mathsf{e}_k\ (i<j<k),\cdots. \tag{1.6}$$

Thus, we often call $\{\mathsf{e}_i\}$ the *generators* of the algebra. It can be readily checked that the dimension of the r-grade subalgebra is

$$\dim(\mathcal{G}_n^r)=\binom{n}{r}.$$

This way, the dimension of the whole algebra is $\dim(\mathcal{G}_n)=2^n$.

Some basic properties of the geometric algebra $\mathcal{G}_n$ are explored below.

Grade of Products

By expanding out multivectors via the computational basis, we observe that the geometric product between them either results in the cancellation of pairs of equal basis vectors via $\mathsf{e}_i\mathsf{e}_i=1$ or results in a wedge product induced by the two orthonormal basis vectors. Given two homogeneous multivectors $A=\langle A\rangle_m$ and $B=\langle B\rangle_n$, their product contains terms of

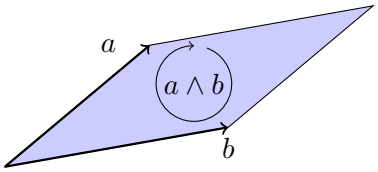
$$AB=\langle AB\rangle_{|m-n|}+\langle AB\rangle_{|m-n|+2}+\cdots+\langle AB\rangle_{m+n-2}+\langle AB\rangle_{m+n}. \tag{1.7}$$

This is a powerful observation and is often utilized with the grade-projection operator to simplify proofs. Moreover, for general multivectors, we denote

$$A\cdot B=\langle AB\rangle_{|m-n|},\ A\wedge B=\langle AB\rangle_{m+n}. \tag{1.8}$$

Bivectors

Multivectors of grade 2 are termed *bivectors* (grade 3 multivectors are called *trivectors*, and so on). They represent, geometrically, a surface encoded with orientation.



As one can see above, the bivector $a \wedge b$ represents the rotating parallelogram.

Pseudoscalars

The highest-grade multivector is called the *pseudoscalar* of the algebra, often denoted by I. In $\mathcal{G}_3$, it has the property of $I^2 = II = (\mathtt{e}_1\mathtt{e}_2\mathtt{e}_3)(\mathtt{e}_1\mathtt{e}_2\mathtt{e}_3) = -1$. Since it is the only basis for the grade 3 elements, all trivectors in $\mathcal{G}_3$ can be represented as νI, representing an oriented volume, where ν is the volume of the element.

A simple property for the pseudoscalar of $\mathcal{G}_3$ is $I = \mathtt{e}_1\mathtt{e}_2\mathtt{e}_3$ commutes with vectors and bivectors, and can be readily checked by analysis in the computational basis.

Duality

For geometric algebra $\mathcal{G}_n$, a duality between its r-grade elements $\mathcal{G}_n^r$ and $(n-r)$-grade elements $\mathcal{G}_n^{n-r}$ can be found:

$$\forall A \in \mathcal{G}_n^r,\ \exists\, B \in \mathcal{G}_n^{n-r} \text{ s.t. } B = \star A = A^\star.$$

B is called the *dual* of A, $\star A$ is read as "(*hodge*) *star* A". For our case of interest, the dual has a really simple relationship with the pseudoscalar of the algebra, namely

$$A^\star = IA. \tag{1.9}$$

The dual of an element represents its *orthogonal complement*. For example, in $\mathbb{R}^3$, the orthogonal complement of the direction $\mathtt{e}_3$ is the bivector plane $\mathtt{e}_1\mathtt{e}_2 = \star(\mathtt{e}_3) = I\mathtt{e}_3$.

Reversion

The reversion of a multivector $M = a_1 a_2 \cdots a_r$ is denoted as

$$M^{\sim} \equiv \tilde{M} = a_r \cdots a_2 a_1. \tag{1.10}$$

It is simply reversing the order of geometric products.

Reflection

The beauty of geometric algebra lies in its geometrical interpretations. Let us look at reflection of a vector a across the direction of a unit vector n, see Figure 1.

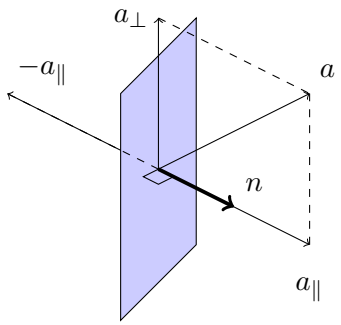


Figure 1: Reflection along a vector direction.

The reflected vector[5] is

$$\begin{aligned} a' &= a - 2a_\parallel = a - 2(a \cdot n)n/|n|^2 \\ &= a - (an + na)n^{-1} = a - ann^{-1} - nan^{-1} \\ &= -nan^{-1} = -nan. \end{aligned}$$

The reflection transformation, however, needs to be modified for general r-grade element:

$$A'_r = (-1)^r n A_r n. \tag{1.11}$$

[5] If n is not normalized, $a' = -nan^{-1}$

Rotation

Two consecutive reflections make up a rotation.

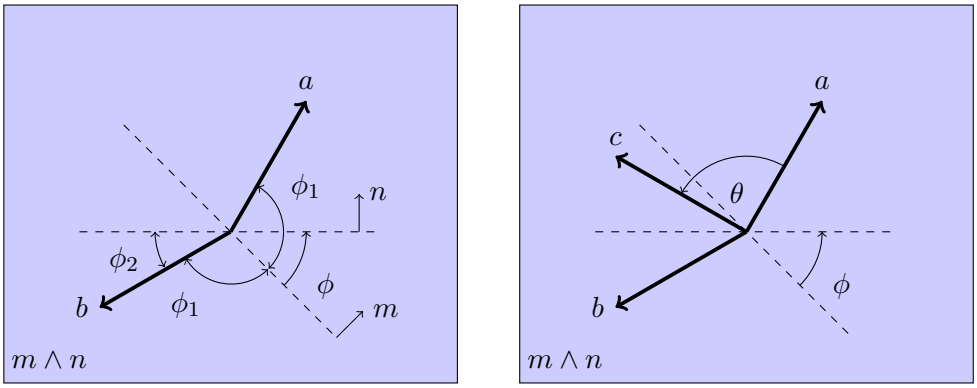


Figure 2: Rotation by two reflections.

As shown above in Figure 2, the vector a is first reflected along the unit vector m to obtain b, and then b is reflected along the unit vector n to obtain c. The rotated angle is

$$\theta = 2\phi,$$

where ϕ is the angle between m and n. Therefore we have

$$c = -n(-mam)n = nmamn = (nm)a(nm)^{\sim}.$$

We can introduce a *rotor*

$$R = nm,$$

with $R\tilde{R} = 1 = \tilde{R}R$. Then the action of rotation reduces to a sandwich product

$$c = Ra\tilde{R}. \tag{1.12}$$

Rotation using rotors is convenient, and the same transformation law works for all elements in the geometric algebra! Since

$$A = a_1 a_2 \cdots a_r \overset{\text{rot.}}{\longmapsto} A' = (Ra_1\tilde{R})(Ra_2\tilde{R})\cdots(Ra_r\tilde{R}) = Ra_1a_2\cdots a_r\tilde{R} = RA\tilde{R}. \tag{1.13}$$

Moreover, let us define a *unit* bivector

$$B = \frac{m \wedge n}{\sin \phi}.$$

It represents the orientation and the plane of rotation. Since $B^2 = -1$, we can rewrite the rotor in exponential form as

$$R = m \cdot n - m \wedge n = \cos \phi - B \sin \phi = \mathrm{e}^{-B\phi} = \mathrm{e}^{-B\frac{\theta}{2}}. \tag{1.14}$$

This exponential representation is defined via the Taylor series expansion, and is useful when we are dealing with the concept of rotation in spaces other than $\mathbb{R}^2$ or $\mathbb{R}^3$.

In general, the rotation of a multivector is the sandwich product between a rotor and its reversion. However, if the rotated 1-vector (grade 1 multivector) a is co-plane with the bivector B, then we can further simplify the equation to

$$\mathrm{e}^{-B\theta/2} a \mathrm{e}^{B\theta/2} = \mathrm{e}^{-B\theta} a = a \mathrm{e}^{B\theta}. \tag{1.15}$$

The equation above is exactly the same as the rotation of complex numbers (or more broadly, quaternionic rotations). Later on, when we see exponentiation of bivectors, it should immediately come to mind as a rotation in the orientation of the bivector plane.

1.5 Other Operations on Multivectors

Trace

Consider a multivector M having an equivalent real $n \times n$ matrix representation M, its trace is defined as

$$\mathrm{tr}[\mathsf{M}] = \mathrm{Tr}\,[M] = n\langle M \rangle. \tag{1.16}$$

If the multivector has an equivalent complex $n \times n$ matrix representation M with unit imaginary denoted as ι, its trace is defined as

$$\mathrm{tr}[\mathsf{M}] = \mathrm{Tr}\,[M] = n\left(\langle M \rangle + \langle M/\iota \rangle \iota\right). \tag{1.17}$$

Unitary Operation

A linear function F is unitary if and only if

$$\mathsf{F}(1)\mathsf{F}^\dagger(1) = 1, \tag{1.18}$$

where $\dagger$ is the *adjoint* operator. In the case of $\mathcal{G}_3$, the adjoint operator is the same as the reversion. However, as we will see in subsection 3.5, it is much more than just reversion in the study of multi-particle geometric algebra.

Exponentiation

For any multivector M, we can define its exponentiation as

$$\mathrm{e}^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}. \tag{1.19}$$

The convergence is guaranteed.

2 Single-Qubit Systems

2.1 Spacetime Algebra

To describe a quantum state, we will need to discuss the properties of *spacetime algebra*, a branch of geometric algebra. Given the orthonormal basis vectors

$$\gamma_0,\ \gamma_1,\ \gamma_2,\ \gamma_3, \tag{2.1}$$

satisfying the signature condition of

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}\left[+1\ , -1\ , -1\ , -1\right], \tag{2.2}$$

we can form an algebra from them. The basis of the algebra consists of:

$1 \times$ grade 0	1
$4 \times$ grade 1	$\gamma_0, \gamma_1, \gamma_2, \gamma_3$
$6 \times$ grade 2	$\gamma_1\gamma_0, \gamma_2\gamma_0, \gamma_3\gamma_0, \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3$
$4 \times$ grade 3	$\gamma_1\gamma_2\gamma_3, \gamma_0\gamma_1\gamma_2, \gamma_0\gamma_1\gamma_3, \gamma_0\gamma_2\gamma_3$
$1 \times$ grade 4	$\gamma_0\gamma_1\gamma_2\gamma_3$

This is known as the *spacetime algebra*, essential in describing qubits. The algebra is often denoted as

$$\mathrm{STA} = \mathcal{G}_{1,3} = \mathcal{G}(1,3),$$

where the subscript 1 represents the 1 positive-norm basis spacetime vector γ_0, and 3 are the 3 negative-norm basis spacetime vectors γ_i $(i = 1, 2, 3)$. From now on, the Greek indices $(\mu, \nu, \cdots)$ represent $0, 1, 2, 3$, and the Latin indices $(i, j, \cdots)$ represent $1, 2, 3$.

2.2 An Algebra for Qubits

To describe quantum states (without relativity effects), we will use a subset of STA: the even subalgebra of spacetime algebra, STA^+. It contains the even-grade spacetime

multivectors. We can construct the *generators* of the subalgebra as

$$\sigma_1 := \gamma_1\gamma_0,\ \sigma_2 := \gamma_2\gamma_0,\ \sigma_3 := \gamma_3\gamma_0, \tag{2.3}$$

they satisfy the signature condition of

$$\sigma_i \cdot \sigma_j = \delta_{ij}, \tag{2.4}$$

where δ_{ij} is the Kronecker delta. The even subalgebra formed consists of the bases of the form:

$$\begin{aligned} &1 \times \text{grade } 0 && 1 \\ &3 \times \text{grade } 1 && \sigma_1, \sigma_2, \sigma_3 \\ &3 \times \text{grade } 2 && I\sigma_1, I\sigma_2, I\sigma_3 \\ &1 \times \text{grade } 3 && I := \sigma_1\sigma_2\sigma_3. \end{aligned}$$

Note that for the bivectors (grade 2 multivectors), we used the duality property of the pseudoscalar to represent them. Under the geometric product operation, we have the following isomorphism:

$$\text{STA}^+ \cong \mathcal{G}_3.$$

Nevertheless, the reversion will be

$$(a + b\sigma_k + cI\sigma_j + dI)^\sim = a - b\sigma_k - cI\sigma_j + dI. \tag{2.5}$$

Because of this difference in reversion (compared to $\mathcal{G}_3$), the change in representation from matrix to geometric algebra should be handled with much care. In the following discussions, when *vectors* or *bivectors* are mentioned, we mean that of STA^+.

2.3 Single Qubit

By using geometric algebra, we can view both a quantum state and the operators on it as elements of the same algebra. Hence, every element, whether it be a vector or a matrix in classical quantum computation, has an equivalent geometric algebra representa-

tion. We denote the isomorphic mapping from classical matrix/bra-ket representation to the geometric algebra representation by

$$\mathsf{M} \overset{\text{GA}}{\longmapsto} M, \tag{2.6}$$

where M is a matrix or a state vector, and M is a multivector. Proceeding on with the already-developed notations by Hestenes and others [DL03], a single qubit state can be represented as

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{bmatrix} \overset{\text{GA}}{\longmapsto} \psi = a^0 + a^k I\sigma_k. \tag{2.7}$$

Here, $a^0, a^1, a^2, a^3 \in \mathbb{R}$, with $|a^0|^2 + |a^1|^2 + |a^2|^2 + |a^3|^2 = 1$, and i is the classical imaginary unit. Moreover, from now on, the Einstein summation convention is used when two repeated indices have one raised and the other lowered; it will be noted when the convention is dropped. We can also rewrite (2.7) as

$$\psi = 1(a^0 + I\sigma_3 a^3) + (-I\sigma_2)(-a^2 + I\sigma_3 a^1). \tag{2.8}$$

It appears as if the unit imaginary i is replaced by $I\sigma_3$, the basis for ψ_1 is 1, and the basis for ψ_2 is $-I\sigma_2$. We observe that a state ψ is an element in the even subalgebra (having elements of only even-grade) of $\mathcal{G}_3$, denoted as $\mathcal{G}_3^+$. Here are some basic examples:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{\text{GA}}{\longmapsto} \psi_{|0\rangle} = 1, \qquad |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \overset{\text{GA}}{\longmapsto} \psi_{|+\rangle} = \frac{1}{\sqrt{2}}(1 - I\sigma_2),$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overset{\text{GA}}{\longmapsto} \psi_{|1\rangle} = -I\sigma_2, \qquad |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \overset{\text{GA}}{\longmapsto} \psi_{|-\rangle} = \frac{1}{\sqrt{2}}(1 + I\sigma_2).$$

This establishes the fact that the multivectors $\{1, I\sigma_1, I\sigma_2, I\sigma_3\}$ are the conventional basis states of the real 4-dimensional even subalgebra corresponding to a two-dimensional complex Hilbert space $\mathcal{H}_2$ with standard computational basis given by $\mathcal{B}_{\mathcal{H}_2} = \{|0\rangle, |1\rangle\}$.

And from now on, when "even subalgebra" is mentioned, it is referred to as $\mathcal{G}_3^+$.

Bloch Vector

The Bloch sphere representation of a qubit used is

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \cos\phi\sin\frac{\theta}{2} + i\sin\phi\sin\frac{\theta}{2} \end{bmatrix}.$$

The equivalent geometric algebra representation is

$$\psi = \cos\frac{\theta}{2} + \sin\phi\sin\frac{\theta}{2}I\sigma_1 - \cos\phi\sin\frac{\theta}{2}I\sigma_2 = \cos\frac{\theta}{2} - B\sin\frac{\theta}{2}.$$

Note that the bivector

$$B = -\sin\phi I\sigma_1 + \cos\phi I\sigma_2 = (\cos\phi - \sigma_1\sigma_2\sin\phi)I\sigma_2 = I\mathrm{e}^{-\sigma_1\sigma_2\phi}\sigma_2.$$

This is a rotated plane, making an angle of ϕ with the $+\sigma_1$ axis. And since

$$B^2 = (-\sin\phi I\sigma_1 + \cos\phi I\sigma_2)^2 = -1,$$

we can rewrite the qubit as

$$\psi = \mathrm{e}^{-B\theta/2} = \mathrm{e}^{-I\sigma_3\phi/2}\mathrm{e}^{-I\sigma_2\theta/2}. \tag{2.9}$$

A pure state is, in fact, a rotor! (The rotation needed to rotate σ_3 to the Bloch vector s, i.e., $\psi\sigma_3\tilde{\psi} = s$.) It is also called a *spinor* in the context of quantum. In some literature, (2.7) is introduced via this Bloch vector-rotor (spinor) equivalence. This idea is further generalized when *density operators* are introduced.

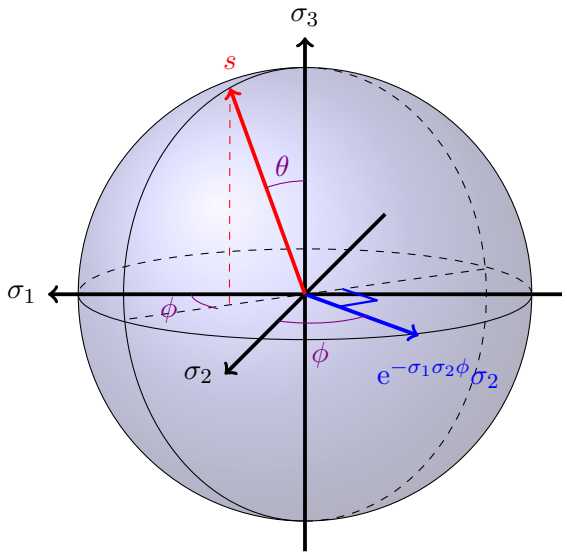


Figure 3: Bloch vector representation of a single qubit.

2.4 Inner Product

Suppose that single qubits $|\psi\rangle, |\phi\rangle$ are represented by vectors a, b. The quantum inner product is defined, in bra-ket notation, as

$$\begin{aligned}\langle\psi|\phi\rangle &= \begin{bmatrix}\psi_1^* & \psi_2^*\end{bmatrix}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \psi_1^*\phi_1 + \psi_2^*\phi_2 \\ &= \begin{bmatrix}a^0 - ia^3 & -a^2 - ia^1\end{bmatrix}\begin{bmatrix}b^0 + ib^3 \\ -b^2 + ib^1\end{bmatrix} \\ &= (a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3) + i(a^0b^3 - a^3b^0 + a^1b^2 - a^2b^1).\end{aligned}$$

Thus we can see that

$$\mathrm{Re}\left\{\langle\psi|\phi\rangle\right\} \overset{\mathrm{GA}}{\longmapsto} \langle\tilde{\psi}\phi\rangle = \langle(a^0 - a^k I\sigma_k)(b^0 + b^l I\sigma_l)\rangle = \sum_{k=0}^{3} a^k b^k. \tag{2.10}$$

And since

$$\langle\psi|\phi\rangle = \mathrm{Re}\left\{\langle\psi|\phi\rangle\right\} - i\mathrm{Re}\left\{\langle\psi|i\phi\rangle\right\},$$

$$\langle\psi|\phi\rangle \overset{\mathrm{GA}}{\longmapsto} \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\sigma_3\rangle I\sigma_3 = \langle\tilde{\psi}\phi\rangle + \langle\tilde{\psi}\phi/I\sigma_3\rangle I\sigma_3. \tag{2.11}$$

This projects out the 1 and $I\sigma_3$ components from $\tilde{\psi}\phi$, we can further denote this quantum projection on a multivector M as $\langle M\rangle_Q$. For even-grade multivectors in three dimensions, we can simply represent it as

$$\langle M\rangle_Q = \frac{1}{2}(M + \sigma_3 M \sigma_3). \tag{2.12}$$

Thus,

$$\langle\psi|\phi\rangle \overset{\text{GA}}{\longmapsto} \langle\tilde{\psi}\phi\rangle_Q. \tag{2.13}$$

Note that when multiplying the resulting inner product (2.11) to a state, the $I\sigma_3$ should be placed on the right of the state acted on, as will be explained in the next section.

2.5 Single-Qubit Operators

In classical matrix representation, we separate the operators from the quantum states, and denote the operators with a hat.

Pauli Matrices

When a state is applied by the Pauli matrix operators $\hat{\sigma}_i$, the geometric algebra equivalent will be

$$\hat{\sigma}_1|\psi\rangle = \begin{bmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{bmatrix} \overset{\text{GA}}{\longmapsto} -a^0 I\sigma_2 + a^1 I\sigma_3 - a^2 + a^3 I\sigma_1 = \sigma_1(a^0 + a^k I\sigma_k)\sigma_3$$

$$\hat{\sigma}_2|\psi\rangle = \begin{bmatrix} a^1 + ia^2 \\ -a^3 + ia^0 \end{bmatrix} \overset{\text{GA}}{\longmapsto} a^0 I\sigma_1 + a^1 + a^2 I\sigma_3 + a^3 I\sigma_2 = \sigma_2(a^0 + a^k I\sigma_k)\sigma_3$$

$$\hat{\sigma}_3|\psi\rangle = \begin{bmatrix} a^0 + ia^3 \\ a^2 - ia^1 \end{bmatrix} \overset{\text{GA}}{\longmapsto} a^0 - a^1 I\sigma_1 + a^2 I\sigma_2 + a^3 I\sigma_3 = \sigma_3(a^0 + a^k I\sigma_k)\sigma_3.$$

The results can be simply[6] written as

$$\hat{\sigma}_i|\psi\rangle \overset{\text{GA}}{\longmapsto} \sigma_i \psi \sigma_3. \tag{2.14}$$

[6] A better way to write the Pauli operators that doesn't introduce the clumsy "right multiplication by σ_3" is introduced in section 5. The current method is still introduced as most literature uses it.

The σ_3 at the end is to keep ψ in the even subalgebra. Moreover, one can easily see that the Pauli matrices $\{\hat{\sigma}_i\}$ are *isomorphic* to $\{\sigma_i\}$, hence the notation. And the usual identities of Pauli matrices are easily obtained via geometric algebra:

$$\hat{\sigma}_i^2 = \hat{\mathsf{I}} \overset{\text{GA}}{\longmapsto} \sigma_i^2 = 1$$
$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = i\hat{\mathsf{I}} \overset{\text{GA}}{\longmapsto} \sigma_1\sigma_2\sigma_3 = I,$$

where $\hat{\mathsf{I}}$ is the 2×2 identity matrix. Another thing we can check is:

$$\boldsymbol{r} = \langle \psi | \boldsymbol{\sigma} | \psi \rangle,$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli matrix vector, and the Bloch vector $\boldsymbol{s} = \boldsymbol{r} \cdot \boldsymbol{\sigma}$. Described in geometric algebra, we have

$$\begin{aligned}\langle \psi | \sigma_k | \psi \rangle \overset{\text{GA}}{\longmapsto} \langle (\tilde{\psi})(\sigma_k \psi \sigma_3) \rangle_Q &= \langle \tilde{\psi}\sigma_k\psi\sigma_3 \rangle - \cancel{\langle (\tilde{\psi}\sigma_k\psi\sigma_3) I\sigma_3 \rangle I\sigma_3} \\ &= \begin{cases} \sin\theta\cos\phi & , \ k = 1 \\ \sin\theta\sin\phi & , \ k = 2 \\ \cos\theta & , \ k = 3. \end{cases}\end{aligned}$$

Using grade-projection operator, the quantum inner product can be readily calculated, without ever actually expanding out the product! The latter projection picks out the $I\sigma_3$ components, it is equivalent to finding the I component of $\tilde{\psi}\sigma_k\psi$. But since $\tilde{\psi}\sigma_k\psi = -(\tilde{\psi}\sigma_k\psi)^{\sim}$, it must only contain vector and bivector components (by (2.5)). Hence, the latter projection is zero. By the cyclic-invariance of 0-grade-projection, the former projection can be rewritten as $\langle \sigma_k \psi \sigma_3 \tilde{\psi} \rangle = \langle \sigma_k \boldsymbol{s} \rangle = \sigma_k \cdot \boldsymbol{s}$, which is the vector inner product between σ_k and the Bloch vector $\boldsymbol{s} = \sin\theta\cos\phi\ \sigma_1 + \sin\theta\sin\phi\ \sigma_2 + \cos\theta\ \sigma_3$.

Multiplication by the Imaginary Unit

And multiplication by the imaginary unit results in

$$i|\psi\rangle \overset{\text{GA}}{\longmapsto} I\psi\sigma_3 = \psi I\sigma_3. \tag{2.15}$$

Having seen how in geometric algebra, we no longer need the imaginary unit, and we only need to work with real numbers. σ_3 is to be expected by our chosen conventional representation of (2.7) and (2.9) [DL03].

The multiplication of a state by the unit imaginary (represented in geometric algebra) $I\sigma_3$ is a sandwich product, hence we often denote it as $\mathsf{J} = I\sigma_3$, and

$$\mathsf{J}\psi = I\psi\sigma_3. \tag{2.16}$$

Hadamard Gate*

Since we have

$$\hat{\mathsf{H}} = \frac{1}{\sqrt{2}}(\hat{\sigma}_1 + \hat{\sigma}_3),$$

we can easily obtain that the Hadamard gate is

$$\hat{\mathsf{H}}|\psi\rangle \overset{\text{GA}}{\longmapsto} \left(\frac{\sigma_1 + \sigma_3}{\sqrt{2}}\right)\psi\sigma_3 := \mathsf{H}\psi\sigma_3. \tag{2.17}$$

2.6 Outer Product

SU(2) Isomorphism

Given the isomorphism between a state vector and a multivector: $|\psi\rangle \overset{\text{GA}}{\longmapsto} \psi$, and the isomorphism between Pauli matrices and multivectors: $\hat{\sigma}_k \overset{\text{GA}}{\longmapsto} \sigma_k$ (with $k = 0$ being the identity transformation 1), an isomorphism between a state vector and an SU(2) matrix can be induced:

$$\begin{aligned}
|\psi\rangle = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \begin{bmatrix}a^0 + ia^3\\ -a^2 + ia^1\end{bmatrix} &\overset{\text{GA}}{\longmapsto} a^0 + a^1 I\sigma_1 + a^2 I\sigma_2 + a^3 I\sigma_3 \\
&\overset{\text{SU}}{\longmapsto} a^0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + ia^1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + ia^2\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + ia^3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \\
&= \begin{bmatrix}a^0 + ia^3 & a^2 + ia^1\\ -a^2 + ia^1 & a^0 - ia^3\end{bmatrix} = \begin{bmatrix}\psi_1 & -\psi_2^*\\ \psi_2 & \psi_1^*\end{bmatrix} := \underline{\Psi}.
\end{aligned}$$

We shall use underlined capital Greek letter to represent the SU(2) equivalent of a state. Also, note the relation between hermitian conjugation and reversion operator:

$$\underline{\Psi}^\dagger \overset{\text{GA}}{\longmapsto} \tilde{\psi} \tag{2.18}$$

$$\hat{\sigma}_k^\dagger \overset{\text{GA}}{\longmapsto} \sigma_k. \tag{2.19}$$

Outer Product

For two qubits $|\psi\rangle$ and $|\phi\rangle$, their outer product, in bra-ket notation, is defined as

$$\begin{aligned}|\psi\rangle\langle\phi| &= \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}\begin{bmatrix}\phi_1^* & \phi_2^*\end{bmatrix} = \begin{bmatrix}\psi_1 & 0 \\ \psi_2 & 0\end{bmatrix}\begin{bmatrix}\phi_1^* & \phi_2^* \\ 0 & 0\end{bmatrix} = \underline{\Psi}\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\left(\underline{\Phi}\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\right)^\dagger \\ &= \underline{\Psi}\underline{E_+}\underline{\Phi}^\dagger = \underline{\Psi}\frac{1}{2}(\mathsf{I} + \hat{\sigma}_3)\underline{\Phi}^\dagger.\end{aligned}$$

Note that E_+ ($\underline{E_+}$) is idempotent and $\underline{E_+} \equiv (1+\hat{\sigma}_3)/2 \equiv \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix} \overset{\text{GA}}{\longmapsto} E_+ = (1+\sigma_3)/2$, thus we have

$$|\psi\rangle\langle\phi| \overset{\text{GA}}{\longmapsto} \frac{1}{2}\psi(1+\sigma_3)\tilde{\phi}. \tag{2.20}$$

Moreover, when $\psi = \phi$, we have the density operator for a pure state:

$$\rho = \psi\left(\frac{1+\sigma_3}{2}\right)\tilde{\psi} = \psi E_+ \tilde{\psi}. \tag{2.21}$$

The density operator can be expanded into a scalar plus a vector s.

$$\rho = \frac{1}{2}(1 + \psi\sigma_3\tilde{\psi}) = \frac{1}{2}(1+s)$$

The *spin vector* of a quantum state is simply the vector component of the density operator multiplied by the reduced Planck's constant:

$$\hat{s} = \frac{\hbar}{2}s.$$

2.7 NMR Spectroscopy*

A qubit or a quantum state evolves dynamically according to the Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t}. \tag{2.22}$$

Let us consider a qubit in a constant magnetic field $\boldsymbol{B}$. The state has a magnetic moment $\hat{\mu}$ proportional to its spin vector $\hat{s}$ that interacts with the magnetic field, namely

$$\hat{\mu} = \gamma\hat{s}. \tag{2.23}$$

γ is the gyromagnetic ratio. The magnetic moment has Hamiltonian (note that the Einstein summation convention is used):

$$\hat{H} = -\hat{\mu}\cdot\boldsymbol{B} = -\frac{1}{2}\gamma\hbar B_k\sigma_k. \tag{2.24}$$

Since

$$i\hat{\sigma}_k|\psi\rangle \xmapsto{\text{GA}} \sigma_k\psi\sigma_3(I\sigma_3) = I\sigma_k\psi,$$

we can replace the Hamiltonian and quantum state in the Schrödinger equation by its geometric algebra equivalence:

$$\dot{\psi} = \frac{1}{2}\gamma B_k I\sigma_k\psi = \frac{1}{2}\gamma I\boldsymbol{B}\psi. \tag{2.25}$$

The solution to this rotor equation is

$$\psi(t) = \mathrm{e}^{\gamma I\boldsymbol{B}t/2}\psi_0. \tag{2.26}$$

Hence the motion of the spin vector $\hat{s} = \frac{\hbar}{2}s = \frac{\hbar}{2}\psi\sigma_3\tilde{\psi}$ is just precessing in the $I\boldsymbol{B}$ plane at an angular velocity of $\omega_0 = \gamma|\boldsymbol{B}|$.

We further consider a time-varying magnetic field in NMR [DL03]: let $S = \mathrm{e}^{-I\sigma_3\frac{\omega t}{2}}$,

$$\boldsymbol{B} = B_1(\cos(\omega t)\sigma_1 + \sin(\omega t)\sigma_2) + B_0\sigma_3 = S(B_1\sigma_1 + B_0\sigma_3)\tilde{S} = S\boldsymbol{B_c}\tilde{S}.$$

The rotor equation of (2.25) simplifies to

$$\tilde{S}\dot{\psi} = \frac{1}{2}\gamma I \boldsymbol{B}_c \tilde{S}\psi. \tag{2.27}$$

We see how the qubit evolves through time:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{S}\psi\right) = \dot{\tilde{S}}\psi + \tilde{S}\dot{\psi} = \frac{1}{2}(\omega I\sigma_3 + \gamma I\boldsymbol{B}_c)\tilde{S}\psi, \\
&\quad \to \psi(t) = \exp\left(-I\sigma_3\frac{\omega t}{2}\right)\exp\left(\frac{1}{2}(\omega_0+\omega)tI\sigma_3 + \frac{1}{2}\omega_1 t I\sigma_1\right)\psi_0,
\end{aligned}$$

where $\omega_0 = \gamma B_0$, $\omega_1 = \gamma B_1$. It can be further simplified to

$$\psi(t) = \mathrm{e}^{-I\sigma_3\omega t/2}\left(\cos(\alpha t/2) + I\hat{\boldsymbol{B}}\sin(\alpha t/2)\right), \tag{2.28}$$

$$\hat{\boldsymbol{B}} = \frac{(\omega_0+\omega)\sigma_3 + \omega_1\sigma_1}{\alpha}, \quad \alpha = \sqrt{(\omega_0+\omega)^2 + {\omega_1}^2}. \tag{2.29}$$

Suppose the magnetic field is switched on at $t = 0$, with the initial state being spin-up, i.e. $\psi_0 = 1$. The probability that the particle is in spin-down state at time t will be

$$\Pr\{\downarrow\} = |\langle\downarrow|\psi(t)\rangle|^2.$$

But we only need to consider terms in the $I\sigma_1$ and $I\sigma_2$ planes, since

$$\begin{aligned}
\langle\downarrow|\psi(t)\rangle &= \langle I\sigma_2\psi\rangle_Q = \langle I\sigma_2\psi\rangle - I\sigma_3\langle I\sigma_1\psi\rangle \\
&= \mathrm{e}^{-I\sigma_3\omega t/2}\,\frac{\omega_1\sin(\alpha t/2)}{\alpha}I\sigma_3.
\end{aligned}$$

Hence the probability of measuring spin-down state is

$$\Pr\{\downarrow\} = \left(\frac{\omega_1\sin(\alpha t/2)}{\alpha}\right)^2.$$

The probability is maximized at $\alpha t = \pi$ and $\omega = -\omega_0 = -\gamma B_0$, this is the *spin resonance condition* in NMR.

3 Multi-Qubit Systems

Not much can be done with a single qubit, though. Here we introduce a new algebra for describing multi-qubit states: the *multiparticle spacetime algebra* (MSTA).

3.1 Multiparticle Spacetime Algebra

By preparing n copies of the STA^+, we can describe an n-particle system. For example, we have n copies of $I\sigma_3$, namely $I\sigma_3^a$ with $a = 1, 2 \cdots, n$. The multivector bases with different superscripts are from different qubits, they *commute* under the geometric product:

$$\sigma_k^i \sigma_l^j = \gamma_k^i \gamma_0^i \gamma_l^j \gamma_0^j = (-1)^2 \gamma_l^j \gamma_0^j \gamma_k^i \gamma_0^i = \sigma_l^j \sigma_k^i.$$

Since the bases commute, all multivectors from different qubits commute. This gives a natural origin to the tensor product in geometric algebra, possible with the introduction of MSTA. From here on, we provide a review of some of the results.

We will be working in the algebra of:

$$\left(\mathrm{STA}^+\right)^{\otimes n} = \left(\mathrm{STA}^+\right)^n \cong \mathcal{G}_3^{\otimes n} = (\mathcal{G}_3)^n,$$

which consists of all the states and operations we needed. However, a quick examination tells us that the dimensions are too large for states and operators:

1. n-particle states has real dimension $\dim_{\mathbb{R}}(\mathcal{H}_2^n) = 2^{n+1}$, whereas the even subalgebra of $\mathcal{G}_3^{+\otimes n} = \left(\mathcal{G}_3^+\right)^n$ has dimension $\dim((\mathcal{G}_3^+)^n) = 4^n$
2. a complex $2^n \times 2^n$ complex matrix has real dimension 2^{2n+1}, the algebra we work in has dimension $\dim((\mathrm{STA}^+)^n) = 2^{3n}$.

These can be fixed by the introduction of *correlators*.

3.2 Correlators

This subsection is to discuss the technicalities of the GA formulation. The presence of extra degrees of freedom are all due to the presence of different unit imaginary numbers from

each copy of particle space. They can be removed by multiplying through the correlators.

Correlator

We first work with the space of operators: let us introduce an idempotent element called the (*operator*) *correlator* [HD02]:

$$C := \prod_{i=1}^{n} \frac{1}{2}(1 - I^1 I^i). \tag{3.1}$$

The superscripts on the multivectors denote which copy of STA they are from.

1. C commutes with all elements of $\mathcal{G}_3^{\otimes n}$
2. $I^a I^b C = -C$ for $1 \leq a, b \leq n$
3. multiplication by C maps all elements of $\mathcal{G}_3^{\otimes n}$ onto $\mathcal{G}_3^{\otimes n}/C$, which is isomorphic to $2^n \times 2^n$ complex matrices[7]
4. $C^2 = 1$, and C is an identity element in the ideal.

With C, we can safely treat all I^i's as a single I, and we generally omit C from our expressions, too.

SU Matrix Correlator

Right multiplication elements in $\left(\mathcal{G}_3^{\otimes n}/C\right)^+$ (remember that states are in the even subalgebra) by

$$E_+ := E_+^1 E_+^2 \cdots E_+^n, \tag{3.2}$$

gives an n-qubit state equivalent SU matrix, with

$$E_\pm^k := \frac{(1 \pm \sigma_3^k)}{2}. \tag{3.3}$$

This is useful in the description of density operators.

[7] The (two-sided) ideal $\mathcal{G}_3^{\otimes n} C$ is isomorphic to the quotient algebra $\mathcal{G}_3^{\otimes n}/C$, hence the notation. An ideal is an algebraic *black hole*, multiplication of any element in the algebra by elements in the ideal results back in the ideal.

State Correlator

Since we want the unit imaginary numbers to be identical:

$$\begin{aligned} I\sigma_3^1 &= I\sigma_3^k \quad (\forall k) \\ -1 &= I\sigma_3^1 I\sigma_3^k \\ 1 &= \frac{1}{2}(1 - I\sigma_3^1 I\sigma_3^k). \end{aligned}$$

Consider the above for all k and we obtain D such that all spinor states satisfy

$$\psi = \psi D, \tag{3.4}$$

i.e. D is inherently encoded into all spinors. D is termed the state correlator, also called the *directional correlator* [HD02] and *n-particle correlator* [CM11]. The idempotent is defined as

$$D := \prod_{i=2}^{n} \frac{1}{2}(1 - I\sigma_3^1 I\sigma_3^i). \tag{3.5}$$

Right multiplication by D reduces the dimension of a general even-grade multivector to 2^{n+1}, hence objects in $(\mathcal{G}_3^{\otimes n}/D)^+$ are the quantum states (spinors/rotors). D can be interpreted as picking out the first qubit space and correlating all the other spaces to this one. We further define

$$J := DI\sigma_3^a \tag{3.6}$$

for all $a = 1, \cdots, n$, and $J^2 = -D$. Right multiplication by J represents multiplication by the unit imaginary in our n-particle system. And since operations are, conventionally, represented as left multiplication, we introduce the notion of the complex structure by

$$i|\psi\rangle \xmapsto{\text{GA}} \mathsf{J}\psi = \psi J = \psi I\sigma_3^a. \tag{3.7}$$

As a final note, the correlators C and D introduced above is often dropped since they are idempotents and always appear in operators and spinors, respectively.

3.3 Pauli Operators on Multiple Qubits*

A two-qubit example on the correlators is given below.

$$D = \frac{1}{2}(1 - I\sigma_3^1 I\sigma_3^2)$$

$$|0\rangle \otimes |0\rangle \overset{\text{GA}}{\longmapsto} (1)(1)D = D, \qquad |0\rangle \otimes |1\rangle \overset{\text{GA}}{\longmapsto} (1)(-I\sigma_2^2)D = -I\sigma_2^2 D,$$
$$|1\rangle \otimes |0\rangle \overset{\text{GA}}{\longmapsto} (-I\sigma_2^1)(1)D = -I\sigma_2^1 D, \qquad |1\rangle \otimes |1\rangle \overset{\text{GA}}{\longmapsto} (-I\sigma_2^1)(-I\sigma_2^2)D = I\sigma_2^1 I\sigma_2^2 D.$$

Given a state $\psi \in [\text{STA}^+]^2/D$, when it is applied by the Pauli matrices, we have

$$\begin{aligned}
\hat{\sigma}_k \otimes \hat{I}|\psi\rangle &\overset{\text{GA}}{\longmapsto} \sigma_k^1(\psi D)\sigma_3^1 = -\sigma_k^1 \psi D I I \sigma_3^1 = -I\sigma_k^1 \psi D I \sigma_3^1 = -I\sigma_k^1 \psi J \\
\hat{I} \otimes \hat{\sigma}_k|\psi\rangle &\overset{\text{GA}}{\longmapsto} -I\sigma_k^2 \psi J \\
\hat{\sigma}_k \otimes \hat{\sigma}_l|\psi\rangle &\overset{\text{GA}}{\longmapsto} -I\sigma_k^1(-I\sigma_l^2 \psi J)J = -I\sigma_k^1 I\sigma_l^2 \psi D.
\end{aligned}$$

This result can be easily generalized to

$$\hat{\mathsf{I}} \otimes \cdots \otimes \hat{\sigma}_k^a \otimes \cdots \otimes \hat{\mathsf{I}}\,|\psi\rangle \overset{\text{GA}}{\longmapsto} \sigma_k^a \psi \sigma_3^a. \tag{3.8}$$

3.4 Inner Products and Observables

Theorem (Inner product of quantum states)

The quantum inner product [ML13] for n-particle states is

$$\langle\psi|\phi\rangle \overset{\text{GA}}{\longmapsto} 2^{n-1}\left(\langle\phi D\tilde{\psi}\rangle - \langle\phi J\tilde{\psi}\rangle \mathsf{J}\right). \tag{3.9}$$

⟨**pf.**⟩ Notice how $DE_+ = DE_+^1$, and since

$$\langle\psi|\phi\rangle = \text{tr}[|\phi\rangle\langle\psi|] \overset{\text{GA}}{\longmapsto} \langle\tilde{\psi}E_+\phi\rangle_Q = \text{Tr}\left[\phi E\right],$$

we thus have

$$\begin{aligned}\mathrm{Tr}\left[\phi E\right] &= 2^n \left(\langle \phi E_+ \tilde{\psi} + \langle \phi E_+ \tilde{\psi} / I\sigma_3^1 \rangle I\sigma_3^1\right) \\ &= 2^n \left(\langle \phi E_- \tilde{\psi} + \langle \phi E_+ \tilde{\psi} I\sigma_3^1 \rangle I\sigma_3^1\right).\end{aligned}$$

Note that

$$\begin{aligned}\langle \phi E_+ \tilde{\psi}\rangle = \langle \phi D E_+ \tilde{\psi}\rangle &= \langle \phi D E_+^1 \tilde{\psi}\rangle \\ &= \frac{1}{2}\left(\langle \phi D \tilde{\psi}\rangle + \cancel{\langle \phi D \sigma_3^1 \tilde{\psi}\rangle}\right) = \frac{1}{2}\langle \phi D \tilde{\psi}\rangle.\end{aligned}$$

The second 0-grade-projection is zero since the terms inside are odd-grade. Similarly,

$$\begin{aligned}\langle \phi E_+ \tilde{\psi} I\sigma_3^1\rangle &= \frac{1}{2}\left(\langle \phi D \tilde{\psi} I\sigma_3^1\rangle + \cancel{\langle \phi D \sigma_3^1 \tilde{\psi} I\sigma_3^1\rangle}\right) \\ \frac{1}{2}\langle \phi D I\sigma_3 \tilde{\psi}\rangle &= \frac{1}{2}\langle \phi D \tilde{\psi}\rangle.\end{aligned}$$

$\tilde{\phi}$ and $I\sigma_3^1$ can switch position since we only care about the 0-grade part. And thus it is proven. ■

The inclusion of D in the real part is not needed, since it is in ψ and ϕ. And we can easily see that the Second 0-grade-projection represents the complex part of the inner product. $\langle \phi D \tilde{\psi}\rangle$ picks out the elements of grade 0, 4, 8, $\cdots$, whereas $\langle \phi J \tilde{\psi}\rangle$ picks out elements of grade 2, 6, $\cdots$.

For observables in the two-particle case, we have

$$\begin{aligned}\langle \psi | \hat{\sigma}_k \otimes \hat{1} | \psi\rangle &\xmapsto{\text{GA}} -2I\sigma_k^1 \cdot (\psi J \tilde{\psi}) \\ \langle \psi | \hat{\sigma}_j \otimes \hat{\sigma}_k | \psi\rangle &\xmapsto{\text{GA}} -2(I\sigma_j^1 I\sigma_k^2) \cdot (\psi D \tilde{\psi}).\end{aligned}$$

All observables are contained in $\psi D \tilde{\psi}$ and $\psi J \tilde{\psi}$, the corresponding coefficients can be obtained by taking inner products with the appropriate combination of operators.

3.5 Adjoint Operation

Let ψ be an n-particle state. Compared with (2.18), the adjoint of ψ is defined as:

$$\psi^\dagger := \left(\prod_{i=1}^{n} \gamma_0^i\right) \tilde{\psi} \left(\prod_{i=1}^{n} \gamma_0^i\right) \tag{3.10}$$

The notation is "dagger" since it is the same as the Hermitian conjugate in matrix representation, satisfying (compare with (2.5))

$$1^\dagger = 1, \quad (\sigma_k^i)^\dagger = \sigma_k^i, \quad (I\sigma_k^i)^\dagger = -I\sigma_k^i, \quad I^\dagger = -I \tag{3.11}$$

and the relation

$$(\psi\phi)^\dagger = \phi^\dagger \psi^\dagger.$$

The conjugate operation relieves us with much of the inconveniences met in previous sections. For states and operators, they satisfy (compare with (2.18))

$$\underline{\Psi}^\dagger \overset{\text{GA}}{\longmapsto} \psi^\dagger \tag{3.12}$$

$$\hat{\sigma}_k^\dagger \overset{\text{GA}}{\longmapsto} \sigma_k^\dagger. \tag{3.13}$$

All reversions we have met previously (including (2.11), (2.20), (2.21), (3.9)) can be changed to adjoint since the reversion and adjoint give the same result when applying on an element of even-grade (in this case, spinors). The benefit of such sudden change of notation is most evident when we enter section 5, discussing the unitarity of operators. For simplicity, we will continue to use $\sim$ in section 3 and 4, but will adapt to $\dagger$ in section 5.

3.6 Contraction

Also known as *partial trace* in matrix representation, contraction removes the information of a particle. Taking the contraction over the i-th particle is equivalent to removing all terms from the observables $\psi D\tilde{\psi}$ and $\psi J\tilde{\psi}$ with contributions in the i-th particle space.

3.7 Schmidt Decomposition

A qubit state can be represented as

$$\psi(\theta,\phi) = \mathrm{e}^{-I\sigma_3\phi/2}\mathrm{e}^{-I\sigma_2\theta/2}. \tag{3.14}$$

For a general two-particle state, its Schmidt decomposition is:

$$\begin{aligned}\psi &= \cos\frac{\alpha}{2}\psi^1(\theta_1,\phi_1)\psi^2(\theta_2,\phi_2)\mathrm{e}^{J\tau/2} + \sin\frac{\alpha}{2}\psi^1(\theta_1,\phi_1)I\sigma_2^1\psi^2(\theta_2,\phi_2)I\sigma_2^2\mathrm{e}^{-J\tau/2} \\ &= \psi^1(\theta_1,\phi_1)\psi^2(\theta_2,\phi_2)\mathrm{e}^{J\tau/2}(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}I\sigma_2^1I\sigma_2^2),\end{aligned} \tag{3.15}$$

where the entanglement angle $\alpha \in [0, \frac{\pi}{4}]$.

We further define the individual rotors in the respective particle space[DL03]:

$$R = \psi(\theta_1,\phi_1)e^{I\sigma_3\frac{\tau}{4}}, \quad S = \psi(\theta_2,\phi_2)e^{I\sigma_3\frac{\tau}{4}},$$

such that (3.15) simplifies to

$$\psi = R^1S^2(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}I\sigma_2^1I\sigma_2^2). \tag{3.16}$$

4 Density Operators

In quantum information processing, any quantum mechanical system can be completely described by its *density operator*. And moreover, as we will see later on, the density operator formulation makes the equations much more elegant and clean.

4.1 Density operator

In (2.20), we have seen that for a single qubit, the pure state density operator is

$$\rho = |\psi\rangle\langle\psi| \xmapsto{\text{GA}} \psi E_+ \tilde{\psi} = \frac{1}{2}\psi(1+\sigma_3)\tilde{\psi} = \frac{1}{2}(1+s), \tag{4.1}$$

where $s = \psi\sigma_3\tilde{\psi}$ is the Bloch vector resulting from (2.9). The result above simply mirrors that of the matrix representation, where s is replaced by $\boldsymbol{r}\cdot\boldsymbol{\sigma}$. And for an ensemble of $\{p_i, \psi_i\}$, we obtain the single qubit mixed state:

$$\rho = \sum_i p_i |\psi\rangle\langle\psi| \xmapsto{\text{GA}} \sum_i p_i \frac{1}{2}(1+s_i) = \frac{1}{2}(1+S),$$

with $S \equiv \sum_i p_i s_i$ being the ensemble-average Bloch vector with length $||S|| \leq 1$, where the real coefficients satisfy $\sum_i p_i = 1$. The MSTA version of the density matrix of n-interacting qubits is [CM11]

$$\rho = \mathbb{E}\left[(\Psi D)E_+(\Psi D)^\sim\right] C = \mathbb{E}\left[\Psi E_+ \tilde{\Psi}\right], \tag{4.2}$$

with $\mathbb{E}[\cdot]$ taking the expected value of the expression with respect to the stochastic state Ψ, D being the state correlator defined in (3.5), $E_+ := E_+^1 E_+^2 \cdots E_+^n$ the product of idempotents $E_+^k := \frac{1}{2}(1+\sigma_3^k)$, and C being the operator correlator that can be omitted without confusion.

Furthermore [ML13], ρ is the n-qubit density operator if the following condition are satisfied:

1. $\rho \in (\mathcal{G}_3^{\otimes n})^+$;
2. ρ is positive semi-definite;
3. $\text{Tr}[\rho] = 1$;

4. $\rho = \rho^\dagger$.

Remark. The definition about adjoint operation ($\rho = \rho^\dagger$) is given by section 3.5.

If the state is a product state, where $\psi = \psi^1\psi^2\cdots\psi^n$, each ψ^i being the state of i-th particle. Then the density operator will be $\rho = \rho^1\rho^2\cdots\rho^n$, with $\rho^i = \rho^i E^i_+ \tilde{\rho}^i$. When the state is an entangled state, it cannot be expressed as products of density operators.

4.2 Operations on Density Operators

Trace

We observe that the density operator for a single-qubit pure state can be viewed as the sum of a scalar and a vector, where the vector part is related to the corresponding rotation of σ_3 by the spinor ψ, which is $s = \psi\sigma_3\tilde{\psi}$. The trace in the matrix representation is twice the scalar part of the corresponding expression in GA, i.e.,

$$\mathrm{Tr}\,[\hat{\rho}] \overset{\mathrm{GA}}{\longmapsto} \mathrm{Tr}\,[\rho] = 2\langle\rho\rangle. \tag{4.3}$$

This is a clear mirroring of the classical matrix trace. We can prove it by simply checking that it applies for the components of a density operator, namely $\hat{\mathbb{I}} \overset{\mathrm{GA}}{\longmapsto} 1$, $\hat{\sigma}_k \overset{\mathrm{GA}}{\longmapsto} \sigma_k$.

For a more general case of n-particle density operator, it associates with a $2^n \times 2^n$ square matrix, with trace defined to be ([Shi11]):

$$\mathrm{Tr}\,[\rho] = 2^n\langle\rho\rangle. \tag{4.4}$$

Normal quantum operation (trace-preserving)

A *normal* operation, known as trace-preserving in the matrix form, is a linear transformation of a density operator that can be written in the operator-sum form:

$$\rho \mapsto N(\rho) \equiv \sum_i X_i \rho X_i^\dagger, \tag{4.5}$$

where Kraus operators $X_i \in \mathcal{G}_3^{\otimes n}/C$ subject to $\sum_i X_i X_i^\dagger = 1$.

Complete positivity

Recalling the definition of the complete positivity, a linear map $N^{A\to B}$ is completely positive if the map $N^{A\to B}\otimes I_R \overset{GA}{\longmapsto} N^{A\to B}\otimes 1_R$ is positive for all extensions $\mathcal{G}_3^R$, where R denotes the reference system.

Contraction (partial trace)

Suppose that we wish to take the contraction on the subsystem m ($\in [1:n]$). In the language of GA [SCH98], it can be expressed as:

$$\mathrm{Tr}_m[\rho] = 2\langle\rho\rangle^q = E_+^m\rho E_+^m + E_-^m\rho E_-^m + \sigma_1^m(E_+^m\rho E_+^m + E_-^m\rho E_-^m)\sigma_1^m. \tag{4.6}$$

As mentioned prior, contraction may also be expressed by dropping all elements in ρ depending on m, and multiply the remaining factor by 2.

Suppose that we have found all of the observables and density matrix terms in a 2-particle state. After applying contraction on it to discard all of the information for the second particle, we obtain

$$\mathrm{Tr}_2[\rho] = \frac{1}{2}(1+S), \tag{4.7}$$

where the Bloch vector is

$$S \equiv \sum_i p_i s_i = \cos(\alpha) R\sigma_3\tilde{R}. \tag{4.8}$$

This vector no longer has unit length, meaning the remaining density operator represents a mixed state. Entanglement with the second particle absorbs some coherence from the first particle.

4.3 Probabilities

Given two normalized states, their overlap probability can be expressed as

$$P(\psi,\phi) = |\langle\psi|\phi\rangle|^2$$

Using (3.9) to calculate the inner product, the overlap probability of n particles is

$$P(\psi,\phi) = 2^{n-2}\langle(\psi D\tilde{\psi})(\phi D\tilde{\phi}) - (\psi J\tilde{\psi})(\phi J\tilde{\phi})\rangle \tag{4.9}$$

4.4 Measurements

Not all of the GA structures are suitable for measurement. [TN08] For example, Cayley–Klein parameters [8] having four real values are often not conserved in time, hence not available for direct measurement in Hilbert spaces. Fortuitously, their compositions can be measured in 3D Euclidean space as rotating vector components, using pulsed NMR [Hav+00] experiments.

From (3.15), we can construct the bivector observable:

$$\begin{aligned}
\psi J\tilde{\psi} &= R^1S^2(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}I\sigma_2^1I\sigma_2^2)J(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}I\sigma_2^1I\sigma_2^2)\tilde{R}^1\tilde{S}^2 \\
&= \frac{1}{2}R^1S^2\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right)(I\sigma_3^1 + I\sigma_3^2)\tilde{R}^1\tilde{S}^2 \\
&= \frac{1}{2}\cos(\alpha)\left((RI\sigma_3\tilde{R})^1 + (SI\sigma_3\tilde{S})^2\right) \\
&= \frac{1}{2}\cos(\alpha)(A_3^1 + B_3^2)
\end{aligned} \quad , \tag{4.10}$$

$$\begin{aligned}
\psi D\tilde{\psi} &= R^1S^2(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}I\sigma_2^1I\sigma_2^2)D(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}I\sigma_2^1I\sigma_2^2)\tilde{R}^1\tilde{S}^2 \\
&= \frac{1}{2}R^1S^2\left(1 - I\sigma_3^1I\sigma_3^2 + \sin(\alpha)(I\sigma_2^1I\sigma_2^2 - I\sigma_1^1I\sigma_1^2)\right)\tilde{R}^1\tilde{S}^2 \\
&= \frac{1}{2}\left(1 - A_3^1B_3^2 + \sin(\alpha)(A_2^1B_2^2 - A_1^1B_1^2)\right)
\end{aligned} \quad , \tag{4.11}$$

where

$$A_k \equiv RI\sigma_k\tilde{R}, \quad B_k \equiv SI\sigma_k\tilde{S}$$

are set for simplicity.

For example, suppose that we have two separable 2-particle states ψ and ϕ with their

[8] $[\psi_1;\psi_2] \in \mathbb{C}^2$ are called *Cayley–Klein parameters*, where $\psi_1 \equiv \cos(\theta/2) - i\sin(\theta/2)(r\cdot\sigma_3), \psi_2 \equiv \sin(\theta/2)(r\cdot\sigma_1) - i\sin(\theta/2)(r\cdot\sigma_2)$.

observables taking forms

$$2\psi J\tilde{\psi} = A^1 + B^2, \quad 2\psi D\tilde{\psi} = 1 - A^1 B^2 \tag{4.12}$$

and

$$2\phi J\tilde{\phi} = \alpha^1 + \beta^2, \quad 2\phi D\tilde{\phi} = 1 - \alpha^1 \beta^2. \tag{4.13}$$

We obtain their overlap probability:

$$\begin{aligned} P(\psi, \phi) &= \frac{1}{4}\langle (1 - A^1 B^2)(1 - \alpha^1 \beta^2) - (A^1 + B^2)(\alpha^1 + \beta^2) \rangle \\ &= \frac{1}{4}(1 + A \cdot \alpha B \cdot \beta - A \cdot \alpha - B \cdot \beta) \\ &= \frac{1}{2}(1 - A \cdot \alpha)\frac{1}{2}(1 - B \cdot \beta). \end{aligned} \tag{4.14}$$

Remark. We observe that the probability is the product of independent 1-particle probabilities. If any of the states is entangled, this result no longer holds.

5 Quantum Gates and Measurements

All quantum gates can be given a direct geometric interpretation (or have a trivial representation) using geometric algebra. More often than not, when we obtain a quantum gate expressed in matrix notation, its operational meaning can only be derived later on. And the same thing applies to measurements. We seek to make their operational meanings as obvious as possible for one newly acquainted with them.

A small theorem is needed before we continue.

Cartan-Dieudonné theorem

Every orthogonal transformation in an n-dimensional symmetric bilinear space can be described as a composition of at most n reflections.

Hence, unitary operations on qubits (represented in real Bloch spheres) are suitable to be described by our standard geometric algebra of $\mathcal{G}_3^{\otimes n}$. Transformations formed by an even number of reflections are rotations; those formed by an odd number of reflections are generic reflections.

On the other hand, measurements are, unexpectedly, closely related to projections. Geometric representations sometimes provides us with a new point of view to old problems.

5.1 Single-Qubit Gates

Here we discuss gates that operates on a single qubit state. Rotational gates rotate the Bloch vector when applied on

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\phi}\sin\frac{\theta}{2}|1\rangle \xmapsto{\text{GA}} \psi = \cos\frac{\theta}{2} + \sin\phi\sin\frac{\theta}{2}I\sigma_1 - \cos\phi\sin\frac{\theta}{2}I\sigma_2,$$

or the equivalent density operator (note the † donotes adjoint instead of reversion)

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \boldsymbol{r}\cdot\boldsymbol{\sigma}) \xmapsto{\text{GA}} \rho = \psi E_+ \psi^\dagger = \frac{1}{2}(1+s),$$

with $s = \boldsymbol{r}\cdot\boldsymbol{\sigma}$ being the Bloch vector. A general unitary operation F satisfies

$$\mathsf{F}(1)\mathsf{F}^\dagger(1) = 1 \tag{5.1}$$

and either

1. composites with the spinor ψ to form a new spinor $\mathsf{F}(\psi)$, or
2. acts on the density operator ρ, which is sometimes equivalent to directly acting on the Bloch vector $s = \psi\sigma_3\psi^\dagger \mapsto \mathsf{F}(\psi)\sigma_3\mathsf{F}^\dagger(\psi)$.

Both approaches yield the same result.

Pauli Matrices

Consider a single qubit, since multiplying by a global phase leaves the qubit unaltered, the application of $\hat{\sigma}_k$ on the state is equivalent to applying $-i\hat{\sigma}_k$.

$$-i\hat{\sigma}_k|\psi\rangle \overset{\text{GA}}{\longmapsto} -I\sigma_k\psi\sigma_3\sigma_3 = \mathrm{e}^{-I\sigma_k\frac{\pi}{2}}\psi. \tag{5.2}$$

Thus, we can readily see that applying a Pauli gate $\hat{\sigma}_k$ rotates the Bloch vector around the σ_k-axis by 180 degrees. Unlike (2.14) with its right multiplication by σ_3, if we multiply a spinor ψ by another spinor $-I\sigma_k$, it remains a spinor, while also retaining its geometrical Bloch vector representation[9].

Phase Gate

The geometric algebra equivalence representation of the phase gate is

$$\hat{\mathsf{R}}_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i\alpha} \end{bmatrix} \overset{\text{GA}}{\longmapsto} \mathsf{R}_\alpha = \mathrm{e}^{-\sigma_1\sigma_2\frac{\alpha}{2}}. \tag{5.3}$$

One can check that

$$\hat{\mathsf{R}}_\alpha|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i(\phi+\alpha)}\sin\frac{\theta}{2}|1\rangle,$$

to prove the result above. We can obviously see that it is a rotation of angle α on the $\sigma_1\sigma_2$ bivector plane, or equivalently, around the σ_3 axis since $I\sigma_3 = \sigma_1\sigma_2$.

[9] If written out explicitly, one would find that multiplication by $-I\sigma_k$ is not the same as (2.14). The inconsistency is removed when one realizes they are only off by a global phase.

Hadamard Gate

From (2.17) and (2.21), we have

$$\hat{\mathsf{H}}\hat{\rho}\hat{\mathsf{H}}^\dagger \xrightarrow{\text{GA}} \mathsf{H}\rho\mathsf{H}^\dagger. \tag{5.4}$$

One can verify that this is indeed a valid operation on the density operator since $\mathsf{H}^2 = 1$. Furthermore, because H is a vector, the result of performing a sandwich product on the Bloch vector s is, in fact, a *reflection along* H *followed by a reflection through the origin.* A reflection through the origin in an n-dimensional space can be generated by n successive reflections along mutually orthogonal directions, in our case $n = 3$. Hence, the Hadamard gate is 4 reflections. But wait! By the Cartan-Dieudonné theorem, we can definitely reduce the Hadamard gate as a single rotation. Consider only the vector component of ρ:

$$\mathsf{H}s\mathsf{H}^\dagger = -I\mathsf{H}s\mathsf{H}^\dagger I = (-I\mathsf{H})s(-I\mathsf{H})^\dagger = \exp\left(-I\mathsf{H}\frac{\pi}{2}\right)\ s\ \exp\left(I\mathsf{H}\frac{\pi}{2}\right).$$

At last, we can identify the transformation by the Hadamard gate as not just a sum of σ_1 and σ_3, but also as a rotation of 180 degrees along the H-axis, which is just the $(\sigma_1+\sigma_3)/\sqrt{2}$ axis. When the Hadamard operator acts on the density operator, we have

$$\mathsf{H}\rho\mathsf{H}^\dagger = \mathrm{e}^{-I\mathsf{H}\frac{\pi}{2}}\ \rho\ \mathrm{e}^{I\mathsf{H}\frac{\pi}{2}}. \tag{5.5}$$

Moreover, we should redefine the Hadamard gate as

$$H = -I\left(\frac{\sigma_1+\sigma_3}{\sqrt{2}}\right) \tag{5.6}$$

to make a spinor after transformation remain a spinor. The transformation rules are $\psi \mapsto H\psi$ and $\rho \mapsto H\rho H^\dagger$. It seems like left multiplication of vectors by I is the right way to go (if the computation involves operation on a spinor).

In general, a simple rotation by an angle α about an axis $\boldsymbol{n} = \sum_i n_i\sigma_i$ ($||\boldsymbol{n}|| = 1$) in a one-particle space is given by

$$\mathrm{e}^{-I\boldsymbol{n}\alpha/2}.$$

Grover Diffusion Operator

The Grover diffusion operator (following the notation from Wikipedia)

$$U_\phi = 2|\phi\rangle\langle\phi| - \hat{\mathbb{I}} \overset{\text{GA}}{\longmapsto} 2 \cdot \frac{1}{2}(1 + s_\phi) - 1 = s_\phi.$$

It is, in fact, still a rotation of angle π about the Bloch vector direction s_ϕ, similar to that of the Hadamard gate. When acting on a spinor, the spinor transforms as

$$\psi \mapsto s_\phi \psi \sigma_3 \text{ or } I s_\phi \psi,$$

and when acting on the density operator, the density operator transforms as

$$\rho \mapsto s_\phi \rho s_\phi = (-I s_\phi)\rho(-I s_\phi)^\dagger.$$

5.2 Multiple-Qubit Gates

The single qubit gates are just a taste of what is yet to come. When multiple particles are introduced, we must include the correlators[SCH98], too. Consider the class of *primitive idempotents* introduced before:

$$E^m_\pm = \frac{1}{2}(1 \pm \sigma^m_3), \tag{5.7}$$

they follow the following rules of:

$$(E^m_\pm)^2 = E^m_\pm, \;\; E^m_\pm E^m_\mp = 0, \;\; E^m_+ + E^m_- = 1, \;\; \sigma^m_3 E^m_\pm = \pm E^m_\pm.$$

Note that when treating $E^m_\pm$ as an operator acting on state ψ, it multiplies as

$$E^m_\pm[\psi] = \frac{1}{2}(\psi) \pm \frac{1}{2}\sigma^m_3(\psi)\sigma^m_3. \tag{5.8}$$

The multiplication rule seems over complicated. But when applied on the density operators, it will become all the more familiar.

Swap Gate

The two-particle swap gate can easily be obtained by the operation exchanging the raised indices.

$$\mathsf{S}^{(i,j)}[\sigma_k^i] = \mathsf{S}^{(j,i)}[\sigma_k^i] = \sigma_k^j.$$

Controlled-NOT (CX) Gate

Since the controlled-NOT gate, in matrix notation, can be decomposed as

$$\mathrm{CX} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have its geometric algebra equivalence, when operating on density operators, as

$$\begin{aligned} \mathrm{X}^{2|1} &= E_+^1 + E_-^1 \sigma_1^2 \\ &= (1 - E_-^1) + E_-^1 \sigma_1^2, \end{aligned} \tag{5.9}$$

or when operating on either spinors or density operators:

$$\begin{aligned} \mathrm{X}^{2|1} &= E_+^1 + E_-^1 I \sigma_1^2 \\ &= (1 - E_-^1) - E_-^1 I \sigma_1^2. \end{aligned} \tag{5.10}$$

It is worth noting that operators on a spinor must be an even-grade object, hence the two versions for $X^{2|1}$. No such requirement is needed when operating on the density operator because of the sandwich product.

Using the representations above, we can immediately tell that the controlled-NOT gate applies identity on the second particle when the first particle is in zero-state and that it applies σ_1^2-gate, i.e. the Pauli X gate, on the second particle when the first particle is in one-state. The idempotents E_+^1 and E_-^1 represent, respectively, the *conditioning* on the zero- or one-state.

Such thing can easily be checked: let $\psi_0 = 1$ and $\psi_1 = -I\sigma_2^1$, then

$$\begin{cases} E_+^1[\psi_0] = 1 = \psi_0 \\ E_+^1[\psi_1] = 0 \end{cases} \qquad \begin{cases} E_-^1[\psi_0] = 0 \\ E_-^1[\psi_1] = -I\sigma_2^1 = \psi_1 \end{cases}.$$

In fact, with this idea, all conditional gates can be easily synthesized. For example, the controlled-Z gate will be

$$\mathrm{Z}^{2|1} = E_+^1 - E_-^1 I\sigma_3^2 = (1 - E_-^1) - E_-^1 I\sigma_3^2. \tag{5.11}$$

The idea of conditioning can easily be extended. If we want the conditional operator on state (in its Bloch vector representation) s and its orthogonal complement $-s$ (with $s^2 = 1$), we would use the operators

$$E_\pm^s = \frac{1 \pm s}{2}.$$

With the "conditioning" property of $E_\pm^m$'s established, we can create all gates and provide them with easy operational interpretations.

When the controlled-NOT gate is applied onto a density operator, the notation will be far simpler:

$$\rho' = \mathrm{X}^{2|1} \rho \mathrm{X}^{2|1\dagger}, \tag{5.12}$$

and is consistent with our previous "sandwich" products. In fact, the controlled-NOT gate can also be written in exponential form [HD02; SCH98]:

$$\mathrm{X}^{2|1} = \begin{cases} (1 - E_-^1) + E_-^1\sigma_1^2 = \mathrm{e}^{I(1-\sigma_1^2)E_-^1\pi/2} \\ (1 - E_-^1) - E_-^1 I\sigma_1^2 = \mathrm{e}^{-I\sigma_1^2 E_-^1\pi/2} \end{cases}. \tag{5.13}$$

The proof is absolutely NOT[10] straightforward, and can be summarized as a useful theorem below.

[10] Pun intended.

Theorem (Exponential form of conditional involutory gates)

The conditional operator is of the form

$$U^{2|1} = E_+^1 + E_-^1 U^2,$$

where U^2 is a unitary gate on the second system conditioned on the first system. If U^2 satisfies $(U^2)^2 = 1$ (involution, self-inverse), it can be rewritten in the exponential form of

$$U^{2|1} = \exp\left(I(1-U^2)E_-^1 \frac{\pi}{2}\right).$$

If it satisfies $(U^2)^2 = -1$, then it can be rewritten in the exponential form of

$$U^{2|1} = \exp\left(U^2 E_-^1 \frac{\pi}{2}\right).$$

⟨**pf.**⟩ The proof for the two expressions are similar, we will show the former here. Note that

$$\begin{aligned}(1-U^2)^2 &= 1 - 2U^2 + (U^2)^2 = 2(1-U^2)\\ \rightarrow (1-U^2)^n &= 2^{n-1}(1-U^2).\end{aligned}$$

Then we have

$$\begin{aligned}\mathrm{e}^{I(1-U^2)E_-^1\frac{\pi}{2}} &= \sum_{n=0}^{\infty} \frac{\left[I(1-U^2)E_-^1\right]^n \left(\frac{\pi}{2}\right)^n}{n!} = 1 + \sum_{n=1}^{\infty} I^n E_-^1 \frac{2^{n-1}(1-U^2)\left(\frac{\pi}{2}\right)^n}{n!}\\ &= 1 + \frac{1}{2}E_-^1(1-U^2)\sum_{n=1}^{\infty} I^n \frac{\pi^n}{n!} = 1 + \frac{1}{2}E_-^1(1-U^2)(\mathrm{e}^{I\pi}-1)\\ &= 1 + \left(\frac{1}{2}E_-^1 - \frac{1}{2}E_-^1 U^2\right)(-2)\\ &= (1-E_-^1) + E_-^1 U^2 = E_+^1 + E_-^1 U^2. \quad \blacksquare\end{aligned}$$

An involutory, unitary operator is either a 180° rotation or a reflection. Though $E_\pm^m$ by itself is not a unitary operator, the combined conditional gates are indeed unitary. The exponential form of representation is easier for us to uncover its geometrical properties, too.

The proof is essentially the same as finding the exponential generator of the respective gates. We can hence expand it to:

Theorem (Exponential form of conditional gate)

Given a unitary operator U^2 on system 2 with exponential generator G^2 (which is self-adjoint), viz.

$$U^2 = \exp G^2, \tag{5.14}$$

the operator conditioned on system 1 can be represented as

$$U^{2|1} = (1 - E^1_-) + E^1_- U^2 = \exp\left(G^2 E^1\right). \tag{5.15}$$

Toffoli Gate

The quantum Toffoli gate takes in three qubit as input, and only reverses the third qubit if the former two are $|1\rangle$'s, else the third is left unchanged. The first two qubit are not altered after being passed through the gate.

Hence we can express the Toffoli gate as:

$$\mathrm{X}^{3|1,2} = \begin{cases} (E^1_+ E^2_+ + E^1_+ E^2_- + E^1_- E^2_+) + E^1_- E^2_- \sigma^3_1 = (1 - E^1_- E^2_-) + E^1_- E^2_- \sigma^3_1 \\ (1 - E^1_- E^2_-) - E^1_- E^2_- I\sigma^3_1 \end{cases}, \tag{5.16}$$

having the respective the exponential form of

$$X^{3|1,2} = \begin{cases} \exp\left(I(1 - \sigma^3_1) E^1_- E^2_- \frac{\pi}{2}\right) \\ \exp\left(-I\sigma^3_1 E^1_- E^2_- \frac{\pi}{2}\right) \end{cases}. \tag{5.17}$$

5.3 Quantum Fourier Transform

Let us define the Hadamard gate for particle i acting on a density operator as

$$\mathsf{H}^i = \frac{1}{\sqrt{2}}(\sigma^i_1 + \sigma^i_3),$$

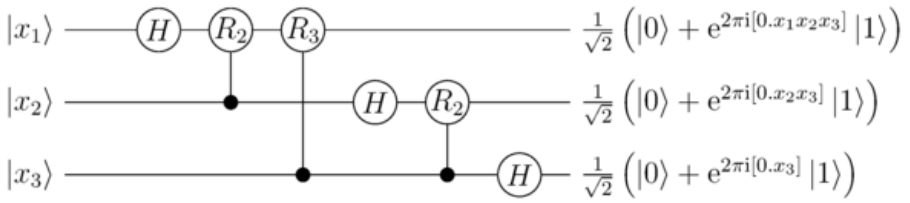


Figure 4: Implementation of quantum fourier transform.

and the phase gate of phase $\frac{2\pi}{2^k}$ for particle i as

$$\mathsf{V}_k^i = \mathrm{e}^{-\sigma_1^i \sigma_2^i \frac{2\pi}{2^k}}.$$

Starting from two qubits, observe that the QFT gate is represented as

$$\mathsf{QFT}_{2^2} = \mathsf{H}^2 \left(E_+^2 + E_-^2 \mathsf{V}_2^1\right) \mathsf{H}^1 = \mathsf{H}^2 \mathrm{e}^{-I\sigma_3^i \frac{2\pi}{2^k} E_-^2} \mathsf{H}^1; \tag{5.18}$$

for three qubits

$$\mathsf{QFT}_{2^3} = \mathsf{H}^3 \left(E_+^3 + E_-^3 \mathsf{V}_2^2\right) \mathsf{H}^2 \left(E_+^3 + E_-^3 \mathsf{V}_3^1\right) \left(E_+^2 + E_-^2 \mathsf{V}_2^1\right) \mathsf{H}^1. \tag{5.19}$$

For a general n-qubit QFT gate, it is represented as an recursive product of one-particle Hadamard gates and two-particle conditional phase gates:

$$\begin{aligned}
\mathsf{QFT}_{2^n} =& \mathsf{H}^n \\
& (E_+^n + E_-^n \mathsf{V}_2^{n-1}) \mathsf{H}^{n-1} \\
& (E_+^n + E_-^n \mathsf{V}_3^{n-1})(E_+^{n-1} + E_-^{n-1} \mathsf{V}_2^{n-2}) \mathsf{H}^{n-2} \\
& \vdots \\
& (E_+^n + E_-^n \mathsf{V}_n^1) \cdots (E_+^3 + E_-^3 \mathsf{V}_3^1)(E_+^2 + E_-^2 \mathsf{V}_2^1) \mathsf{H}^1 \\
=& \prod_{l=n}^{1} \prod_{k=n}^{l+1} \left(E_+^k + E_-^k \mathsf{V}_{k-l+1}^l\right) \mathsf{H}^l.
\end{aligned} \tag{5.20}$$

Note that the product is non-commutative, hence there is an unconventional switch of upper and lower bound in the product. Each qubit, say, register l, is applied sequentially with

the gates of

$$(E_+^n + E_-^n \mathsf{V}_{n-l+1}^l)(E_+^{n-1} + E_-^{n-1}\mathsf{V}_{n-l}^l)\cdots(E_+^{l+1} + E_-^{l+1}\mathsf{V}_2^l)\mathsf{H}^l.$$

In the scenario of a phase estimation algorithm, the input state is either $|0\rangle$ or $|1\rangle$, so H^l simply rotates it to $|+\rangle$ or $|-\rangle$, then the conditional phase gates rotate the Bloch vector on the σ_1^l-σ_2^l plane, viz. the plane normal to the $|0\rangle$-$|1\rangle$ direction. Equivalently, we can first apply rotations on the σ_2^l-σ_3^l plane, then flip it using H^l. Hence, let us define a new phase gate

$$Q_k^i = \mathrm{e}^{-\sigma_2^i \sigma_3^i \frac{2\pi}{2^k}}, \tag{5.21}$$

and the QFT gate can be easily rewritten as

$$\mathsf{QFT}_{2^n} = \prod_{l=n}^{1} \mathsf{H}^l \prod_{k=n}^{l+1} \left(E_+^k + E_-^k \mathsf{Q}_{k-l+1}^l\right). \tag{5.22}$$

In this case, the Hadamard gates for all registers are applied at the end. Hence forth, we can even remove the Hadamard gates entirely in a phase estimation scenario if the oracle also creates phase kickback on the σ_2^l-σ_3^l plane. Or even better, if the conditioning of $E_\pm^k$ can be changed to conditioning on states of $|\pm\rangle$, the Hadamard gates can be removed without modifying the oracle, with input state of $|+\rangle^{\otimes n}$. This provides us with ways to modify the implementation of QFT given future advances in hardware of quantum computing.

Lastly, we can also express QFT in its exponential form:

$$\mathsf{QFT}_{2^n} = \prod_{l=n}^{1} \exp\left(-I\sigma_3^l \sum_{k=l+1}^{n} \frac{2\pi}{2^{k-l+1}} E_-^k\right) \mathsf{H}^l. \tag{5.23}$$

In most literature, QFT is often written in the form of a mapping $|x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_y \mathrm{e}^{2\pi i \frac{x}{2^n} y} |y\rangle$ or as a $2^n \times 2^n$ matrix. But here we can write it as an operator on its own, while retaining the mapping-like representation.

5.4 Projection-Valued Measurements

For a single qubit, the simplest projection-valued measurements (PVMs) are rank-1 projections

$$\hat{\Pi}_i = |i\rangle\langle i| = \frac{1}{2}\left(\hat{\mathsf{I}} + \boldsymbol{r}_i \cdot \boldsymbol{\sigma}\right),$$

projecting the quantum state $|\psi\rangle$ onto the $|i\rangle$ direction (with $|i\rangle \overset{\text{GA}}{\longmapsto} \phi_i$). In geometric algebra, we simply obtain:

$$\begin{aligned}\Pr\{i\} = |\langle i|\psi\rangle|^2 \overset{\text{GA}}{\longmapsto} \left|\langle\phi_i^\dagger\psi\rangle_Q\right|^2 &= \left|\langle\phi_i^\dagger\psi\rangle + \langle\phi_i^\dagger\psi/I\sigma_3\rangle I\sigma_3\right|^2 \\ &= \langle\phi_i^\dagger\psi\rangle^2 - \langle\phi_i^\dagger\psi/I\sigma_3\rangle^2.\end{aligned}$$

Another approach is from density operators: denote $s_i \equiv \boldsymbol{r}_i \cdot \boldsymbol{\sigma}$,

$$\Pr\{i\} = \mathrm{Tr}\left[\hat{\Pi}_i\hat{\rho}\right] = \frac{1}{2}(1 + \boldsymbol{r}_i \cdot \boldsymbol{r}) \overset{\text{GA}}{\longmapsto} \Pr\{i\} = \frac{1}{2}(1 + s_i \cdot s). \tag{5.24}$$

We shall work with density operators here since they can easily be translated to random measurements (POVMs). For n-particle case, in geometric algebra, we also have

$$\Pr\{i\} = \mathrm{Tr}\left[\Pi_i\rho\right] = 2^n\langle\Pi_i\rho\rangle.$$

5.5 Positive Operator-Valued Measurements

Similar to its matrix representation, POVMs are a collection of positive semi-definite operators $\{\Pi_i\}$ that satisfy the completeness relation. And by positive semi-definiteness of a geometric algebraic element, it means that for any density operator ρ,

$$\mathrm{Tr}\left[\rho\Pi_i\right] \geq 0.$$

6 Conclusion

After the introduction of basic operations in geometric algebra, including reflection, rotation, reversion, trace and unitarity, we introduced the STA as our algebraic structure for qubits. We further expanded STA to MSTA to describe multiparticle systems. We first dealt with the dimensionality problem of the geometric algebra in quantum computation by the inclusion of correlators. Then we discussed the geometric-algebra-equivalence to Pauli matrix operations, inner products, Hermitian conjugate. Next, formulations on density is developed. A density operator consists of a scalar part and a vector part, where the latter is its Bloch vector. Lastly, a detailed guide to gates and measurements in the GA formulation is given. The omnipresent sandwich product on density operators showed us how unitary operations are related to rotations on the Bloch sphere. Furthermore, the idea of conditioning operations is described. Theorems on how to write conditioning gates in exponential forms are also given.

Due to the limited time frame, not a whole lot can be covered. Hence, we mainly focused on the unifying the notations and ideas found in past studies. The use of geometric algebra in quantum computation originated from the product operator formulation of NMR spectroscopy, but we believe the benefits provided by geometric algebra should be much more. In fact, future researches can focus on:

1. what roles the norm of multivectors play in quantum information theory. An often used norm is
$$||M|| = \sqrt{\langle M\tilde{M}\rangle}.$$
What about other grade-projection operators? Can GA provide new definitions of distance measures with geometrical meaning?

2. The unique algebraic structures in the wedge product and geometric product may provide us with new ways to encode information.

3. Nowadays, geometric algebra is utilized in control theory, signal processing and computer graphics. Perhaps the natural links between these field with quantum information and computation can be found using geometric algebra.

References

Book Sources

[DL03] Chris Doran and Anthony Lasenby. Geometric Algebra for Physicists. Cambridge University Press, 2003.

[NC10] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2010.

Paper Sources

[SCH98] Shyamal S. Somaroo, David G. Cory, and Timothy F. Havel. "Expressing the operations of quantum computing in multiparticle geometric algebra". In: Physics Letters A 240 (1-2 1998), pp. 1–7. DOI: https://doi.org/10.1016/S0375-9601(98)00010-3.

[Vla99] Alexander Yu Vlasov. "Quantum gates and Clifford algebras". In: arXiv preprint quant-ph/9907079 (1999).

[Hav+00] Timothy F Havel et al. "Geometric algebra methods in quantum information processing by NMR spectroscopy". In: Advances in geometric algebra with applications (2000), pp. 281–308.

[HD02] Timohty F. Havel and Chris J.L. Doran. "Geometric Algebra in Quantum Information Processing". In: Contemporary Mathematics 305 (2002), pp. 81–100.

[Ima08] Isamu Imahori. "Clifford algebras and geometry of entanglement". In: (2008).

[TN08] Victor I Tarkhanov and Michael M Nesterov. "Geometric information in eight dimensions vs. quantum information". In: Quantum Informatics 2007 7023 (2008), pp. 155–168.

[VTF08] José David M Vianna, Marcelo AS Trindade, and Marco César Barbosa Fernandes. "Algebraic criteria for entanglement in multipartite systems". In: International Journal of Theoretical Physics 47 (2008), pp. 961–970.

[WW08] Stephanie Wehner and Andreas Winter. "Higher entropic uncertainty relations for anti-commuting observables". In: Journal of Mathematical Physics 49.6 (2008), p. 062105.

[CM11] Carlo Cafaro and Stefano Mancini. "A Geometric Algebra Perspective on Quantum Computational Gates and Universality in Quantum Computing". In: (2011). DOI: https://doi.org/10.1007/s00006-010-0269-x.

[Shi11] D. S. Shirokov. "Concepts of trace, determinant and inverse of Clifford algebra elements". In: (2011). DOI: 10.48550/arXiv.1108.5447.

[ML13] Nolmar Melo and Carlile Lavor. "A Clifford algebra of signature (n, 3n) and the density operators of quantum information theory". In: Advances in Applied Clifford Algebras 23 (2013), pp. 143–152.

[GB18] M. Gregorič and N.S. Mankoč Borštnik. "Quantum gates and quantum algorithms with Clifford algebra technique". In: (2018). DOI: https://doi.org/10.48550/arXiv.0801.3201.

[PV20] Anupam Prakash and Antonios Varvitsiotis. "Correlation matrices, Clifford algebras, and completely positive semidefinite rank". In: Linear and Multilinear Algebra 68.5 (2020), pp. 1039–1056.

[TFV20] Marco AS Trindade, Sergio Floquet, and J David M Vianna. "A general formulation based on algebraic spinors for the quantum computation". In: International Journal of Geometric Methods in Modern Physics 17.14 (2020), p. 2050206.

[Zhu21] Huangjun Zhu. "Hiding and masking quantum information in complex and real quantum mechanics". In: Physical Review Research 3.3 (2021), p. 033176.

[HNV22] Jaroslav Hrdina, Aleš Návrat, and Petr Vašík.
"Quantum computing based on complex Clifford algebras".
In: Quantum Information Processing 21.9 (2022), p. 310.

[TRF22] Marco AS Trindade, Vinicius NL Rocha, and S Floquet. "Clifford Algebras, Quantum Neural Networks and Generalized Quantum Fourier Transform".
In: arXiv preprint arXiv:2206.01808 (2022).

Other Sources

[sud20] sudgylacmoe. A Swift Introduction to Geometric Algebra. 2020.
URL: https://youtu.be/60z_hpEAtD8.